# Some equivalence results for a bargaining set in finite economies

Javier Hervés-Estévez
RGEA. Universidad de Vigo.
e-mail: javiherves@uvigo.es

Emma Moreno-García
Universidad de Salamanca. IME
e-mail: emmam@usal.es

**Abstract.** We give a notion of bargaining set for finite economies using Aubin's (1979) veto and show its coincidence with the set of Walrasian allocations, providing a discrete approach to the characterization of competitive equilibria obtained by Mas-Colell (1989) for continuum economies.

We also study how the restriction on the formation of coalitions affects the bargaining set. In the last part of the work, using our equivalence result along with some known characterizations of Walrasian allocations, we state additional interpretations of the bargaining set.

**JEL Classification:** D51, D11, D00.

**Keywords:** Bargaining sets, coalitions, core, veto mechanism.

* We are especially grateful to Carlos Hervés-Beloso for his helpful comments and suggestions. We are also indebted to Joao Correia-da-Silva, Marta Faias, and the participants in the European Workshop on General Equilibrium Theory (EWGET 2013, 2014 and 2015), in the V Workshop on Economic Analysis held in Naples in January, 2014, and in the UECE Lisbon Meetings 2014: Game Theory and Applications, where previous versions of this work have been presented.

** This work is partially supported by Research Grant SA072U16 (Junta de Castilla y León) and ECOBAS (Xunta de Galicia).

# 1 Introduction

Aumann and Maschler (1964) introduced the concept of bargaining set, containing the core of a cooperative game. The main idea is to inject a sense of credibility and stability to the veto mechanism, hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. Thus, only objections without counter-objections are considered as credible or justified, and consequently, blocking an allocation becomes more difficult.

This original concept of bargaining set was later adapted to atomless economies by Mas-Colell. Under conditions of generality similar to those required in Aumann's (1964) core-Walras equivalence theorem, Mas-Colell (1989) showed that the bargaining set and the competitive allocations coincide for continuum economies.

In the finite economy framework the core strictly contains the set of Walrasian allocations. Debreu and Scarf (1963) formalized the Edgeworth's (1881) conjecture showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated a sufficiently large number of times. This result yields the definition of Edgeworth equilibrium[1] for an economy with a finite number of agents as an attainable allocation whose $r$-fold repetition belongs to the core of the $r$-fold replica of the original economy, for any positive integer $r$; it can also be defined as an attainable allocation which cannot be blocked by a coalition with rational rates of participation. The veto system proposed by Aubin (1979) allows participations of the agents with any weight in the real unit interval[2]; the corresponding core is a limit notion of Edgeworth equilibrium and equals the set of Walrasian allocations.

In this paper we build upon the Edgeworth equilibrium notion and provide a concept of bargaining set based on the blocking mechanism in the sense of Aubin. It is appropriate to remark that, outside the framework of pure exchange economies, the Aubin veto has been also used by Yang, Liu and Liu (2011) to extend the original bargaining set to Aubin bargaining sets for games which they refer to as convex cooperative fuzzy games. Shortly after, Liu and Liu (2012)

[1]The concept of Edgeworth Equilibrium was defined by Aliprantis, Brown and Burkinshaw (1987). See also Florenzano (1990).

[2]When the weights are rational numbers the Aubin's veto system is the blocking mechanism in the replicated economies (see Remark in page 6).

gave a modification of the previous extension and obtained both existence and equivalence results with other cooperative solutions. It can be then said that we extend and adapt the notions of bargaining sets recently provided by Yang, Liu and Liu (2011) and Liu and Liu (2012) for (transferable utility) cooperative games to finite exchange economies.

Our first main theorem states that the set of Walrasian allocations coincides with this Aubin bargaining set, providing a finite approach to the characterization obtained by Mas-Colell (1989) of competitive allocations. The connection of our result with the related literature can be summarized in the following table:

| | | |
|---|---|---|
| Atomless economies | core-Walras equivalence. (Aumann, 1964) | Mas-Colell's bargaining set-Walras equivalence. (Mas-Colell, 1989) |
| Finite economies | Debreu-Scarf's (1963) core convergence result and its limit version (Aubin, 1979) | (Aubin) bargaining set -Walras equivalence. This paper: Theorem 4.1. |

Equivalence results for Walrasian equilibria.

Our result (and Mas-Colell's) implicitly requires the formation of all coalitions both in the objection and counter-objection processes. It is usually argued that the costs arising from forming a coalition are not at all negligible, and this idea leads us to study the possibility of restricting the formation of coalitions by assuming that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. This question was addressed (for continuum economies) by Schjødt and Sloth (1994), who showed that when one restricts the coalitions participating in objections and counter-objections to those whose size is arbitrarily small, then the Mas-Colell bargaining set becomes strictly larger than the original one. Moreover, Hervés-Estévez and Moreno-García (2015) showed that, in order to obtain the Mas-Colell bargaining set in atomless economies, it is not possible to restrict coalitions in the objection process, independently of the kind of restriction we consider.

The present work shows that, both for objections and counter-objections, the participation rates of the agents can be restricted to those arbitrarily small without changing the bargaining set. However, we also show that this does not hold if we consider parameters close enough to the complete participation. In addition, we prove that the participation rates in the counter-objection system can be restricted to rational numbers, which is the veto power we get when the economy is enlarged via replicas.

In the last part of this paper, we try to make the best use of our results by recasting in terms of the bargaining set some characterizations of the Walrasian allocations already present throughout the literature. First, we focus on results by Hervés-Beloso, Moreno-García and Yannelis (2005a, 2005b) that, in particular, characterize Walrasian allocations as those that are not blocked by the coalition formed by all the agents in a collection of perturbed economies. Then, we follow Hervés-Beloso and Moreno-García (2009), who showed that Walrasian equilibria can be identified by using a non-cooperative two-player game. Both equivalence theorems constitute now additional characterizations of the bargaining set for finite economies.

The rest of the work is structured as follows. In Section 2 we collect notation, preliminaries and definitions. In Section 3, we provide the definition of bargaining set following the veto mechanism in the sense of Aubin. In Section 4, we show our main equivalence result and a characterization of justified objections via Walrasian objections. Section 5 elaborates on the possibility of restricting the coalitions that are allowed to form and still get the bargaining set. In Section 6, specific equivalence theorems for Walrasian equilibrium are presented as further characterizations of the bargaining sets.

## 2 Preliminaries

Let $\mathcal{E}$ be an exchange economy with $n$ agents, who trade $\ell$ commodities. Each consumer $i$ has a preference relation $\succsim_i$ on the set of consumption bundles $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[3] and strict monotonicity. This implies that preferences are represented by utility functions $U_i, i \in N = \{1, \dots, n\}$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ denote the endowments of consumer $i$. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation $x$ is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation $x$ is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell - 1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian

[3]The convexity we require is the following: If a consumption bundle $z$ is strictly preferred to $\hat{z}$ so is the convex combination $\lambda z + (1 - \lambda)\hat{z}$ for any $\lambda \in (0, 1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

equilibrium for the economy $\mathcal{E}$ is a pair $(p, x)$, where $p$ is a price system and $x$ is a feasible allocation such that, for every agent $i$, the bundle $x_i$ maximizes the utility function $U_i$ in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation $y$ is said to be attainable or feasible for the coalition $S$ if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. The coalition $S$ blocks the allocation $x$ if there exists an allocation $y$ which is attainable for $S$, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some member $j$ in $S$. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked by any coalition of agents.

It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibrium and that any Walrasian allocation belongs to the core (in particular, it is efficient).

We will also use the fact that, regarding Walrasian equilibria, a finite economy $\mathcal{E}$ with $n$ consumers can be identified with a continuum economy $\mathcal{E}_c$ with $n$-types of agents as we specify below.

Consider the finite economy $\mathcal{E}$ and let $\mathcal{E}_c$ be the continuum economy where the set of agents is $I = [0, 1] = \bigcup_{i=1}^n I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$. Every $t \in I_i$ has endowments $\omega(t) = \omega_i$ and preference $\succsim_t = \succsim_i$, that is, all the consumers in $I_i$ are of the same type $i$. In this particular case, $x = (x_1, \ldots, x_n)$ is a Walrasian allocation in the finite economy $\mathcal{E}$ if and only if the step function $f_x$ (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in the continuum economy $\mathcal{E}_c$. Moreover, if $f$ is a competitive allocation in $\mathcal{E}_c$, the allocation $x^f = (x_1^f, \ldots, x_n^f)$ given by $x_i^f = \frac{1}{n} \int_{I_i} f(t) d\mu(t)$ is a Walrasian allocation in the finite economy $\mathcal{E}$.

Mas-Colell (1989) stated a notion of bargaining set for continuum economies and showed that it characterizes the competitive allocations. The definition of Mas-Colell's bargaining set is as follows:

An objection to the allocation $f$ in the economy $\mathcal{E}_c$ is a pair $(S, y)$, where $y$ is an attainable allocation for the coalition $S$, such that $y(t) \succeq_t f(t)$ for every $t \in S$ and $\mu(\{t \in S | y(t) \succ_t f(t)\}) > 0$. A counter-objection to the objection $(S, y)$ is a pair $(T, z)$, where $z$ is an attainable allocation for the coalition $T$, such that $z(t) \succ_t y(t)$ for every $t \in T \cap S$ and $z(t) \succ_t f(t)$ for every $t \in T \setminus S$.

An objection which cannot be counter-objected is said to be justified. The

Mas-Colell bargaining set of a continuum economy $\mathcal{E}_c$, denoted by $B_{MC}(\mathcal{E}_c)$, is the set of feasible allocations for which there is no justified objection.

# 3 A bargaining set for finite economies

To characterize the Walrasian equilibria in terms of the core, Aubin (1979) enlarges the veto power of coalitions in finite economies in order to block every non-Walrasian allocation.

An allocation $x$ is blocked in the sense of Aubin by the coalition $S$ via the allocation $y$ if there exist participation rates $\alpha_i \in (0,1]$, for each $i \in S$, such that (i) $\sum_{i\in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$, and (ii) $y_i \succsim_i x_i$, for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. The Aubin core of the economy $\mathcal{E}$, denoted by $C_A(\mathcal{E})$, is the set of all feasible allocations which cannot be blocked in the sense of Aubin. Under the assumptions previously stated, Aubin (1979) showed that $C_A(\mathcal{E}) = W(\mathcal{E})$.

**Remark.** When the participation rates are rational numbers, the veto mechanism in the sense of Aubin is the veto system in sequence of replicated economies. To be precise, if the parameters defining the participations rates of each member in a blocking coalition $S$ are rational numbers, then there are natural numbers $a_i, i \in S$ and $r \geq \max\{a_i, i \in S\}$, such that $\alpha_i = a_i/r$ for every $i \in S$. That is, we can say that the blocking coalition is formed by $a_i$ agents of type $i$.

An *Aubin objection* to $x$ in the economy $\mathcal{E}$ is a pair $(S, y)$, where $S$ is a coalition that blocks $x$ via $y$ in the sense of Aubin. An *Aubin counter-objection* to the objection $(S, y)$ is a pair $(T, z)$, where $T$ is a coalition and $z$ is an allocation defined on $T$, for which there exist $\lambda_i \in (0,1]$ for each $i \in T$, such that:

(i) $\sum_{i\in T} \lambda_i z_i \leq \sum_{i\in T} \lambda_i \omega_i$,

(ii) $z_i \succ_i y_i$ for every $i \in T \cap S$ and

(iii) $z_i \succ_i x_i$ for every $i \in T \setminus S$.

From now on, every time we are in a finite economy framework and write block, objection, counter-objection, or any other concept related with a veto system, we refer to those notions in the sense of Aubin unless stated otherwise.

**Definition** 3.1 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy $\mathcal{E}$, denoted by $B(\mathcal{E}$, if it has no justified objection. A justified objection is an objection that has no counter-objection.*

Note that $C_A(\mathcal{E}) \subseteq B(\mathcal{E})$.

# 4 An equivalence result

To state our main theorem, we prove a previous result.

**Lemma** 4.1 *Let $x$ be an allocation in $\mathcal{E}$. If $(S,g)$ is a justified objection (in the sense of Mas-Colell) to $f_x$ in the associated $n$-types continuum economy $\mathcal{E}_c$, then $(\bar{S},\bar{g})$ is a justified objection to $x$ in the finite $\mathcal{E}$, where $\bar{S} = \{i \in N \mid \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.*

**Proof:** Let us assume that $f_x$ is objected by $(S,g)$ meaning that: $\int_S g(t)d\mu(t) \le \int_S \omega(t)d\mu(t)$, $g \succsim_t f_x$ for every $t \in S$ and $\mu\left(\{t \in S | g \succ_t f_x\}\right) > 0$. Let $S_i = S \cap I_i$ and $\bar{S} = \{i \in N | \mu(S_i) > 0\}$. Since $S$ blocks $f_x$ via $g$, we have that there exists a type $k \in N$ and a set $A \subset S_k = S \cap I_k$, with $\mu(A) > 0$, such that $g(t) \succ_k f_x$, for every $t \in A$.

Let $\bar{g}$ be the allocation given by $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$. Then, by convexity of the preferences, we have $\bar{g}_i \succsim_i x_i = f_x(t)$ for every $t \in S_i = S \cap I_i$ and $i \in \bar{S}$; and $\bar{g}_k \succ_k x_k = f_x(t)$ for every $t \in S_k$.[4] Thus, $(\bar{S},\bar{g})$ is an objection *à la Aubin* to the allocation $x$ in the economy $\mathcal{E}$, since we have that: *(i)* $\sum_{i \in \bar{S}} \mu(S_i)\bar{g}_i \le \sum_{i \in \bar{S}} \mu(S_i)\omega_i$, *(ii)* $\bar{g}_i \succsim_i x_i$ for every $i \in \bar{S}$ and *(iii)* there exists $k \in \bar{S}$ such that $\bar{g}_k \succ_k x_k$.

Assume that the objection $(\bar{S},\bar{g})$ has a counter-objection $(\bar{T},z)$, that is, there exists $\{\lambda_i\}_{i \in \bar{T}}$ with $\lambda_i \in (0,1]$ for every $i \in \bar{T}$, such that: *(i)* $\sum_{i \in \bar{T}} \lambda_i z_i \le \sum_{i \in \bar{T}} \lambda_i \omega_i$, *(ii)* $z_i \succ_i \bar{g}_i$ for every $i \in \bar{T} \cap \bar{S}$ and *(iii)* $z_i \succ_i x_i$ for every $i \in \bar{T} \setminus \bar{S}$.

If $\bar{T} \cap \bar{S} = \emptyset$ then, in the associated continuum economy $\mathcal{E}_c$, any coalition $T = \bigcup_{i \in \bar{T}} T_i \subset I$ with $\mu(T_i) = \lambda_i$, counter-objects the objection $(S,g)$ via the allocation $f_z$ given by $f_z(t) = z_i$ for every $t \in T_i$. Otherwise (i.e., $\bar{T} \cap \bar{S} \neq \emptyset$), from the previous condition *(ii)* we can deduce that for every $i \in \bar{T} \cap \bar{S}$, there exists $A_i \subset S_i$ with $\mu(A_i) > 0$, such that $z_i \succ_i g(t)$ for every $t \in A_i$. This is again

[4] See the Lemma in García-Cutrín and Hervés-Beloso (1993) for further details.

a consequence of the convexity property of preferences. Let $a = \min\{\mu(A_i), i \in \bar{T} \cap \bar{S}\}$ and take $M$ large enough such that $\alpha_i = \frac{\lambda_i}{M} \leq a$ for every $i \in \bar{T}$.

Consider a coalition $T \subset I$ in the continuum economy $\mathcal{E}_c$ with $T = \cup_{i \in \bar{T}} T_i$, such that $T_i \subset A_i$, if $i \in \bar{T} \cap \bar{S}$; $T_i \subset I_i$, if $i \in \bar{T} \setminus \bar{S}$ and $\mu(T_i) = \alpha_i$, for every $i \in \bar{T}$. Then, defining the step function $h$ as $h(t) = z_i$ if $t \in T_i$, we have that: *(i)* $\int_T h(t) d\mu(t) = \sum_{i \in \bar{T}} \alpha_i z_i \leq \sum_{i \in \bar{T}} \alpha_i \omega_i = \int_T \omega(t) d\mu(t)$, *(ii)* $h(t) \succ_i g(t)$ for every $t \in T_i$ with $i \in \bar{T} \cap \bar{S}$; and *(iii)* $h(t) \succ_i x_i = f_x(t)$ for every $t \in T_i$ with $i \in \bar{T} \setminus \bar{S}$.

Note that *(ii)* and *(iii)* mean $h(t) \succ_t g(t)$ for every $t \in T \cap S$ and $h(t) \succ_t f_x(t)$ for every $t \in T \setminus S$, respectively. In other words, we have constructed a counter-objection $(T, h)$ for the objection $(S, g)$, which concludes the proof.

Q.E.D.

In particular, we can conclude that if $(S, g)$ is a justified objection (in the sense of Mas-Colell) to $f_x$ in $\mathcal{E}_c$, then so is $(S, \hat{g})$, where $\hat{g}(t) = \bar{g}_i$ for every $t \in S_i = S \cap I_i$ and every $i \in \bar{S}$.[5]

**Theorem** 4.1 *The bargaining set of the finite economy $\mathcal{E}$ coincides with the set of Walrasian allocations.*

**Proof:** Since the Aubin core coincides with the set of Walrasian allocations for the economy $\mathcal{E}$, we have that any Walrasian allocation has no objection in the sense of Aubin and therefore belongs to the bargaining set of $\mathcal{E}$.

Let us show that $B(\mathcal{E}) \subseteq W(\mathcal{E})$. Consider an allocation $x \in B(\mathcal{E})$ and the step function $f_x(t) = x_i$ if $t \in I_i$, which is a feasible allocation in the associated $n$-types continuum economy $\mathcal{E}_c$. It suffices to show that $f_x$ belongs to the Mas-Colell bargaining set of $\mathcal{E}_c$.[6] Let us assume that $f_x$ is blocked by the coalition $S$ via the allocation $g$ in $\mathcal{E}_c$ and that $(S, g)$ is a justified objection to $f_x$ in the sense of Mas-Colell. By Lemma 4.1 we can ensure that $(\bar{S}, \bar{g})$ is a justified objection to $x$ in $\mathcal{E}$, where $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S} = \{i \in N \mid \mu(S \cap I_i) > 0\}$. This is in contradiction to the fact that $x \in B(\mathcal{E})$.

Q.E.D.

[5]We stress that when preferences are not strictly convex we cannot ensure that every justified objection in the $n$-types continuum economy has the equal-treatment property. However, the Lemma 4.1 ensures that given a justified objection in $\mathcal{E}_c$, there is also an equal-treatment justified objection.

[6]This is so because the Mas-Colell bargaining set of $\mathcal{E}_c$ equals the set of competitive allocations (Mas-Colell, 1989), which is also equivalent to the core (Aumann, 1964), and $f_x$ is competitive in $\mathcal{E}_c$ if and only if $x$ is Walrasian in $\mathcal{E}$.

Enlarging the set of coalitions has a double effect. On the one hand, objecting is easier and allows for more justified objections which, in turn, would make the bargaining set smaller. On the other hand, counter-objecting is also easier, which would eliminate more objections, making it more difficult for the equivalence to hold. As we have already pointed out, there is still another effect that comes from the aforementioned fact that if a consumer participates in both an objection and counter-objection, then an improvement is required in the counter-objection with respect the objection for such an agent, independently of the participation rate in the objection. The aggregate effect is therefore not clear, which highlights the fact that the equivalence result is not straightforward.

Next we characterize justified objections as Walrasian objections, which was also done by Mas-Colell (1989) for continuum economies.

**Definition** 4.1 *Let $x$ be an allocation in the finite economy $\mathcal{E}$. An (Aubin) objection $(S, y)$ to $x$ is said to be Walrasian if there exists a price system $p$ such that (i) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (ii) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$.*

We remark that, under the assumptions of monotonicity and strict positivity of the endowments, we know that $p \gg 0$, and therefore conditions *(i)* and *(ii)* above can be written as follows: $v \succ_i y_i$ implies $p \cdot v > p \cdot \omega_i$, for $i \in S$ and $v \succ_i x_i$ implies $p \cdot v > p \cdot \omega_i$ for $i \notin S$.

Observe that the notion of Walrasian objection in the finite economy $\mathcal{E}$, in contrast to Mas-Colell's for the continuum case, does not depend on the rates of participation of the members in the coalition that objects an allocation; what does become important is the set of consumers who are involved in the objection.

**Theorem** 4.2 *Let $x$ be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to the allocation $x$ is justified if and only if it is a Walrasian objection.*

**Proof:** Let $(S, y)$ be an objection *à la Aubin* to $x$. Assume $(T, z)$ is a counter-objection in the sense of Aubin to $(S, y)$. Then, there exist coefficients $\lambda_i \in (0, 1]$ for each $i \in T$, such that: $\sum_{i \in T} \lambda_i z_i \leq \sum_{i \in T} \lambda_i \omega_i$; $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$. Since $(S, y)$ is a Walrasian objection at prices $p$ we have that $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \cap S$ and $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \setminus S$. This implies $p \cdot \sum_{i \in T} \lambda_i z_i > p \cdot \sum_{i \in T} \lambda_i \omega_i$, which contradicts that $z$ is attainable by $T$ with weights $\lambda_i, i \in T$. Thus, we conclude that $(S, y)$ is a justified objection.

To show the converse, let $(S, y)$ be a justified objection to $x$ and let $a = (a_1, \ldots, a_n)$ be an allocation (not necessarily feasible) such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every consumer $i$ define $\Gamma_i = \{z \in \mathbb{R}^\ell | z + \omega_i \succsim_i a_i\} \bigcup \{0\}$ and let $\Gamma$ be the convex hull of the union of the sets $\Gamma_i, i \in N$.

Let us show that $\Gamma \bigcap (-\mathbb{R}^\ell_{++})$ is empty. Assume that $\delta \in \Gamma \bigcap (-\mathbb{R}^\ell_{++})$. Then, there is $\lambda = (\lambda_i, i \in N) \in [0,1]^n$, with $\sum_{i=1}^n \lambda_i = 1$, such that $\delta = \sum_{i=1}^n \lambda_i z_i \in \Gamma$. This implies that the coalition $T = \{j \in N \mid \lambda_j > 0\}$ counter-objects $(S, y)$ via the allocation $\hat{z}$ where $\hat{z}_i = z_i + \omega_i - \delta$ for each $i \in T$. Indeed, $\sum_{j \in T} \lambda_j \hat{z}_j = \sum_{j \in T} \lambda_j \omega_j$. Moreover, since $z_i \in \Gamma_i$ for every $i \in T$ and $\delta \ll 0$, by monotonicity of preferences, $\hat{z}_i \succ_i y_i$ for every $i \in T \cap S$ and $\hat{z}_i \succ_i x_i$ for every $i \in T \setminus S$. This is a contradiction.

Thus, $\Gamma \bigcap (-\mathbb{R}^\ell_{++}) = \emptyset$, which implies that 0 is a frontier point of $\Gamma$. Therefore, there exists a hyperplane that supports $\Gamma$ at 0. That is, there exists a price system $p$ such that $p \cdot z \geq 0$ for every $z \in \Gamma$. This means that $p \cdot v \geq p \cdot \omega_i$, if $v \succsim_i a_i$. Therefore, we conclude that $(S, y)$ is a Walrasian objection.

Q.E.D.

**Remark:** The previous characterization highlights one of the most important differences between the finite and the continuum frameworks: if $(S, y)$ is a justified objection to $x$ and $y_i \succ_i x_i$, this does not imply full participation of the agents of type $i$. This is in contrast to Mas-Colell's notion for which if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection.

# 5 Restricting coalition formation

In this section, we assume that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible, and study the consequences that this assumption has with regard to the bargaining set solution.

To this end, we consider that a coalition $S$ is defined by the rates of participation of its members, which is given by a vector $\lambda_S = (\lambda_i, i \in S) \in (0,1]^{|S|}$, where $|S|$ denotes the cardinality of $S$.

Consider that for each coalition $S$ the participation rates are restricted to a subset $\Lambda_S \subset [0,1]^{|S|}$. Let us denote by $B_\Lambda(\mathcal{E})$ (respectively $B^\Lambda(\mathcal{E})$) the bargaining set where a coalition $S$ can object (respectively counter-object) only with participation rates in $\Lambda_S$. When the set of coalitions is restricted in the objection (respectively counter-objection) process, it becomes harder to block an allocation (respectively to counter-object an objection) and thus we have $B^{\Lambda}(\mathcal{E}) \subseteq B(\mathcal{E}) \subseteq B_{\Lambda}(\mathcal{E})$. In addition, if $\Lambda, \widehat{\Lambda}$ are such that $\Lambda_S \subseteq \widehat{\Lambda}_S$ for every coalition $S$, then $B^{\Lambda}(\mathcal{E}) \subseteq B^{\widehat{\Lambda}}(\mathcal{E})$ but $B_{\widehat{\Lambda}}(\mathcal{E}) \subseteq B_{\Lambda}(\mathcal{E})$. Therefore, restricting the set of coalitions which are able to object enlarges the bargaining set, whereas restricting the coalition formation in the counter-objection mechanism diminishes the bargaining set. This is so because when not all the coalitions can take part in the bargaining mechanism, on the one hand blocking is harder but, on the other hand, it is easier for an admissible objection to become credible or justified.

The next result is related to the remark on the core of atomless economies stated by Schmeilder (1972). Given $\delta \in (0, 1]$, let $\delta$-$B(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where the participation rate of any agent in any coalition, both in the objection and counter-objection procedure, is restricted to be less or equal than $\delta$.

**Theorem** 5.1 *All the $\delta$-bargaining sets are equal and coincide with the bargaining set in the finite economy $\mathcal{E}$. That is, $\delta$-$B(\mathcal{E}) = B(\mathcal{E})$, for every $\delta \in (0, 1]$.*

**Proof:** Let an allocation $y$ be attainable for a coalition $S$ with participation rates $\lambda_i, i \in S$. That is, $\sum_{i \in S} \lambda_i y_i \leq \sum_{i \in S} \lambda_i \omega_i$. It suffices to note that there exists $(\alpha_i, i \in S)$, with $\alpha_i \leq \delta$ for every $i \in S$ such that $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$. To see this, let $M$ be large enough so that $\alpha_i = \lambda_i / M \leq \delta$, for every $i \in S$. Thus, the same allocation $y$ is also attainable for the same coalition $S$ with participation rates arbitrarily small. The same reasoning holds for the case of both objections and counter-objections.

Q.E.D.

The above result is in contrast to the work by Schjødt and Sloth (1994), marking a contrast between Mas-Collel's bargaining set for continuum economies and our finite approach.

Symmetrically to Schmeidler's (1972) and Grodal's (1972) core characterizations for atomless economies, Vind (1972) showed that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions. The next example shows that such restriction does not produce a similar effect for the bargaining set we address.

**Example 1.** Let $\mathcal{E}$ be an economy with two consumers who trade two commodities, $a$ and $b$. Both agents have the same preference relation represented by the utility function $U(a,b) = ab$, and both are initially endowed with one unit of each commodity. Let us consider the feasible allocation $x$ which assigns the bundle $x_1 = (2,2)$ to the individual 1 and the bundle $x_2 = (0,0)$ to individual 2. The allocation $x$ does not belong to the bargaining set (it does not belong to the core and it is not a Walrasian allocation). In fact, $x$ is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Moreover, every objection $(\{2\},(1,1))$, with any $\lambda \in (0,1]$, has no counter-objection *à la Aubin* and, therefore, is justified.

Next we state a similar example showing that we cannot state such a restriction in the counter-objecting mechanism either.

**Example 2.** Let $\mathcal{E}$ be an economy with three consumers who trade two commodities, $a$ and $b$. All the agents have the same preference relation represented by the utility function $U(a,b) = ab$, and are initially endowed with one unit of each commodity. Let us consider the feasible allocation $x$ which assigns the bundle $x_1 = (3,3)$ to individual 1 and the bundle $x_2 = x_3 = (0,0)$ to individuals 2 and 3. The allocation $x$ is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Note also that $(\{3\},(1,1))$ is a counter-objection to the objection $(\{2\},(1,1))$ . However, there is no counter-objection to $(\{2\},(1,1))$ if all the participation rates are required to be, for instance, larger than $1/2$.[7] To see this, assume that $\{1,2,3\}$ counter-objects, with weights $\lambda_i, i = 1,2,3$. Given the preference relations, we can conclude that $3\lambda_1 + \lambda_2 < \lambda_1 + \lambda_2 + \lambda_3$. We obtain a contradiction with the fact that $\lambda_1, \lambda_3 \in (1/2,1]$.

To finish this section, we consider yet another restriction for the participation rates of the agents in coalitions. As the following equivalence states, it turns out that the bargaining set is entirely characterized when the participation rates of agents in coalitions involved in counter-objections are rational numbers.

**Theorem** 5.2 *Let $B^Q(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where only rational numbers are allowed as participation rates in the counter-objection process. Then, $B^Q(\mathcal{E}) = B(\mathcal{E})$.*

**Proof:** Let $x$ be a feasible allocation and $(S,y)$ an objection to $x$. Let $(T,z)$ be

[7]The same remains true if the parameters are required to be larger than any number in $(1/2,1)$.

a counter-objection to $(S, y)$. This means that there exist coefficients $\alpha_i$, $i \in T$, such that (i) $\sum_{i \in T} \alpha_i z_i = \sum_{i \in T} \alpha_i \omega_i$ and (ii) $z_i \succ_i y_i$ for every $i \in T \cap S$, and $z_i \succ_i x_i$ for every $i \in T \setminus S$.

For every natural $k \in \mathbb{N}$, we define $a_i^k$, $i \in T$, as the smallest integer greater than or equal to $k\alpha_i$. Let us denote $z_i^k = \dfrac{k\alpha_i}{a_i^k}(z_i - \omega_i) + \omega_i$. Since $\lim_{k \to \infty} z_i^k = z_i$ for every $i \in T$, by continuity of preferences, we have that $z_i^k \succ_i y_i$ for every $i \in T \cap S$ and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all $k$ large enough.

By construction, we have $\sum_{i \in T} a_i^k (z_i^k - \omega_i) = 0$. Denoting $q_i^k = \frac{a_i^k}{\sum_{i \in T} a_i^k}$ we obtain (i) $\sum_{i \in T} q_i^k z_i^k = \sum_{i \in T} q_i^k \omega_i$ and (ii) $z_i^k \succ_i y_i$ for every $i \in T \cap S$, and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all $k$ large enough.

Q.E.D.

# 6 Additional characterizations

In this section we pick up two different ways of identifying Walrasian allocations and recast them in terms of bargaining sets.

First, let us consider a feasible allocation $x = (x_1, \ldots, x_n)$ in the economy $\mathcal{E}$. Following Hervés-Beloso, Moreno-García and Yannelis (2005a, 2005b), we define a family of economies denoted by $\mathcal{E}(a, x)$, $a = (a_1, \ldots, a_n) \in [0, 1]^n$, which coincide with $\mathcal{E}$ except for the endowments that, for each agent $i \in N$, are defined by $\omega_i(a, x) = a_i x_i + (1 - a_i)\omega_i$. An allocation (feasible or not ) is said to be dominated in the economy $\mathcal{E}$ if it is blocked by the grand coalition $N$. In the previously cited works it was proved that, under the assumptions we have considered, an allocation $x$ is Walrasian in the economy $\mathcal{E}$ if and only if it is not dominated in any perturbed economy $\mathcal{E}(a, x)$.

The essence of the second characterization of Walrasian equilibrium that we recast for bargaining sets differs substantially from the previous ones. It follows a non-cooperative game theoretical approach and provides insights into the mechanism through which the bargaining process is conducted.

Given the finite economy $\mathcal{E} = (\mathbb{R}_+^\ell, \succsim_i, \omega_i, i \in N)$, let us define an associated game $\mathcal{G}$ as follows. There are two players. The strategy sets for the players are:

$S_1 = \{\ x = (x_1, ..., x_n) \in \mathbb{R}_+^{\ell n}$ such that $x_i \neq 0$ and $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i\}$.

$S_2 = \{(a, y) \in [\alpha, 1]^n \times \mathbb{R}_+^{\ell n}$ such that $\sum_{i=1}^n a_i y_i \leq \sum_{i=1}^n a_i \omega_i\}$, where $\alpha$ is a real number such that $0 < \alpha < 1$.

Given a strategy profile $s = (x,(a,y)) \in S_1 \times S_2$, the payoff functions $\Pi_1$ and $\Pi_2$, for player 1 and 2, respectively, are defined as $\Pi_1(x,(a,y)) = \min_i\{U_i(x_i) - U_i(y_i)\}$ and $\Pi_2(x,(a,y)) = \min_i\{a_i\ (U_i(y_i) - U_i(x_i))\}$.

Note that if $\Pi_2(x,(a,y)) > 0$, then the allocation $x$ is blocked via $y$ by the big coalition being $a_i$ the participation rate of each consumer $i$. Actually, player 2 gets a positive payoff if and only if the big coalition objects in the sense of Aubin the allocation proposed by player 1.

As an immediate consequence of our bargaining-Walras equivalence and the aforementioned characterizations of Walrasian allocations, we obtain further equivalences in terms of bargaining sets.

**Corollary** 6.1 *Let $x$ be a feasible allocation in the economy $\mathcal{E}$. The following statements are equivalent:*

*(i)* *$x$ belongs to the bargaining set of $\mathcal{E}$*

*(ii)* *$x$ not dominated in any economy $\mathcal{E}(a,x)$.*

*(iii)* *$(x,(\mathbf{b},x))$ with $\mathbf{b}_i = b$, for every $i = 1,\ldots,n$, ( for instance $(x,(\mathbf{1},x))$ ) is a Nash equilibrium for the game $\mathcal{G}$.*

An alternative way of stating part of the above result is: *The allocation $x$ has a justified objection (equivalently, a Walrasian objection) in the economy $\mathcal{E}$ if and only if $x$ is blocked by the grand coalition in some economy $\mathcal{E}(a,x)$.*

# References


Aliprantis, C.D., Brown, D.J., Burkinshaw, O. (1987): Edgeworth Equilibria. Econometrica 55, 1109–1137.

Aubin, J.P. (1979): Mathematical methods of game economic theory. North-Holland, Amsterdam, New York, Oxford.

Aumann, R.J. (1964): Markets with a continuum of traders. Econometrica 32, 39–50.

Aumann, R., Maschler, M. (1964): The bargaining set for cooperative games, in: M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in game theory (Princeton University Press, Princeton, NJ), 443–447.

Debreu, G., Scarf, H. (1963): A limit theorem on the core of an economy. International Economic Review 4 (3), 235–246.

Edgeworth, F.Y. (1881): Mathematical Psychics. Paul Kegan, London.

Florenzano, M. (1990): Edgeworth equilibria, fuzzy core, and equilibria of a production economy without ordered preferences. Journal of Mathematical Analysis and Applications 153, 18-36.

García-Cutrín, J., Hervés-Beloso, C. (1993): A discrete approach to continuum economies. Economic Theory 3, 577-584.

Grodal, B. (1972): A second remark on the core of an atomless economy. Econometrica 40, 581–583.

Hervés-Beloso, C., Moreno-García, E. and N.C. Yannelis (2005a): An equivalence theorem for a differential information economy. Journal of Mathematical Economics 41 844?856.

Hervés-Beloso, C., Moreno-García, E., Yannelis, N.C. (2005b): Characterization and incentive compatibility of Walrasian expectations equilibrium in infinite dimensional commodity spaces. Economic Theory, 26, 361-381.

Hervés-Beloso, C., Moreno-García, E. (2009): Walrasian analysis via two-player games. Games and Economic Behavior 65, 220–233.

Hervés-Estévez, J., Moreno-García, E. (2015): On restricted bargaining sets. International Journal of Game Theory 44, 631–645.

Liu, J., Liu, X. (2012): Fuzzy extensions of bargaining sets and their existence in cooperative fuzzy games. Fuzzy sets and systems 188, 88–101.

Mas-Colell, A. (1989): An equivalence theorem for a bargaining set. Journal of Mathematical Economics 18, 129–139.

Schjødt, U., Sloth, B. (1994): Bargaining sets with small coalitions. International Journal of Game Theory 23, 49–55.

Schmeidler, D. (1972): A remark on the core of an atomless economy. Econometrica 40, 579–580.

Vind, K. (1972): A third remark on the core of an atomless economy. Econometrica 40, 585–586.

Yang, W., Liu, J., Liu, X. (2011): Aubin cores and bargaining sets for convex cooperative fuzzy games. International Journal of Game Theory 40, 467–479.