Learning by doing, endogenous discounting and economic development

Stéphane Bouché *

Universidad de Vigo, ECOBAS and RGEA

Abstract

In this paper, we focus on a growth model where the discount rate is decreasing in capital accumulation and endogenous growth is made possible through learning by doing. Knowledge accumulation being a by-product of gross investment. In such a model, the utility function has to be restricted to take positive values implying that the elasticity of marginal utility is lower than one. The presence of endogenous discounting generates the existence of a steady-state of stagnation which can be interpreted as a poverty trap. In the case of long run growth, the fact that the elasticity of marginal utility is lower than one implies the existence of two asymptotic balanced growth paths. The one with the higher growth rate being a saddle point while the one with the lower growth rate not being a saddle point. We also study the optimal solution which is characterized by a unique balanced growth path. The policy consists as usual in subsidizing investment in order to internalize the externality.

Keywords: Endogenous growth, learning by doing, endogenous discounting, indeterminacy, poverty trap

JEL classification: D90, O11, O39, O41

*Correspondance address: Stéphane Bouché. Facultad de Ciencias Económicas y Empresariales. Universidad de Vigo. Campus as Lagoas-Marcosende. 36310 Vigo. Phone: +34986812535. Email: sbouche@uvigo.es

1 Introduction

The objective of the present paper is to introduce endogenous discounting in a growth model where learning by doing is the engine of sustained per capita growth in the long run. It is well known that the standard practice is to use constant discount rates in growth theory and in macroeconomics in general. Our objective consists in showing that the introduction of endogenous discounting will allow us to shed light on some important stylized facts of the growth process:
1) The world distribution of income is now bimodal with a group of countries experiencing growth while another one seems to be stuck at lower levels of development (Quah, 1996; Jones, 1997; Beaudry et al., 2005).
2) Among countries experiencing continuous growth, differences in terms of growth rates and income levels can be observed (see, for example, Acemoglu, 2008).
3) The discount rate does not seem to be constant but decreasing in capital accumulation (Lawrance, 1991; Samwick, 1998; Fielding and Torres, 2009).
On theoretical grounds, there is a strand of growth theory focusing on relaxing the assumption of a constant discount rate that is used in most dynamic models. The first study goes back to Uzawa (1968) but this idea has not received a lot of attention until the work of Obstfeld (1990). Both papers suppose that the discount rate depends on the individual consumption level of the agent. This implies that when taking their decisions, agents internalize the impact of an increase in consumption on the discount rate (see also Drugeon, 1998; Das, 2003). A different approach explored in the literature supposes that the discount rate depends on so called social variables (such as average or aggregate consumption, capital,...) that are not under the control of the agent. In this case, endogenous discounting is seen as an additional externality. The works of Shi (1999), Schmitt-Grohé and Uribe (2003) or Meng (2006) enter in this category. Among the frameworks where agents internalize endogenous discounting, some authors have chosen to focus on other variables than consumption. Le Kama and Schubert (2007) choose environmental quality while Schumacher (2009, 2011) and Strulik (2012) focus on the individual capital stock. The last two authors suppose that the discount rate is decreasing in capital accumulation. It is this latter approach that we are going to follow in the present work. The choice of capital as the argument of the discount function is mainly driven by empirical evidence. Capital can directly be related to wealth accumulation which in turn is positively correlated with lower mortality rates and thus lower discount rates (Fielding and Torres, 2009). Moreover, authors such as Lawrance (1991) and Samwick (1998) have found that the discount rate is decreasing in capital

and wealth in panel data analysis. One important implication of endogenous discounting is that the sign of the utility function matters and can give rise to opposite results depending on a particular choice (see Schumacher, 2011). As we will see, when the discount rate is decreasing in capital accumulation, a negative utility function would imply that welfare is decreasing in capital. An outcome which does not make sense from an economic point of view and leads us to choose a utility function that can only take positive values. However, this assumption implies the breakdown of the standard Mangasarian second order conditions which require a negative utility function. We thus derive appropriate second order conditions and choose our functional forms in accordance.

The idea that learning by doing is an important determinant of the knowledge of workers goes back to Arrow (1962). According to him, a good index of the stock of knowledge is cumulated investment. Arrow used a vintage approach in his work and Levhari (1966) showed that the results could be extended to any homogeneous production function of the first degree. The idea was integrated in the neoclassical growth model with optimizing agents by Sheshinski (1967) and Romer (1986). The latter could demonstrate that his model may generate long run growth with a fully endogenous growth rate. In the present paper, we will follow the approach of Greiner and Semmler (1996) and Greiner (2003), which supposes that one unit of investment does not have the same impact on the formation of physical and knowledge capital. Moreover, their depreciation rates will also be different. This prevents those two variables to be merged into a single one as it is the case in the model of Romer.

We choose to focus on a learning by doing process since it is reasonably tractable but also because empirical evidence has shown that investment is clearly associated with positive externalities (see, DeLong and Summers, 1991; Hamilton and Monteagudo, 1998).

The combination of endogenous discounting with our learning by doing formulation will generate the possibility of poverty traps as well as both local and global indeterminacy. Poverty traps are a potential explanation concerning the fact that some economies seem to be trapped at lower levels of development (Azariadis, 1996). Local indeterminacy implies a continuum of dynamic paths leading to the same balanced growth path while global indeterminacy implies the existence of multiple balanced growth paths (Benhabib and Farmer, 1994). Indeterminacy is thus a potential way to explain why economies with similar fundamentals might save and grow at different rates in both the short and the long run.

A brief account of the results is as follows. We first argue that the utility function should always take positive values in this framework and derive

appropriate sufficiency conditions in this case. We then choose specific functional forms in accordance with this assumption. We derive next necessary and sufficient conditions for the existence of a saddle-path stable steady-state which can be interpreted as a poverty trap. In the case of long run growth, we show that our model generates global indeterminacy with the existence of two asymptotic balanced growth paths. The one with the higher growth rate being a saddle point while the one with the lower growth rate not being a saddle point. The latter can exhibit different dynamic behaviors such as instability, local indeterminacy or a possible limit-cycle trough the existence of a Hopf bifurcation. The study of the optimal outcome shows that there is a unique asymptotic balanced growth path in this case. An appropriate policy giving incentives to increase investment can ensure that all economies converge toward balanced growth.
The structure of the paper is the following: section 2 introduces the model and derives appropriate sufficiency conditions in the case of a utility function that can only take positive values. Section 3 studies the existence of both steady-state equilibria and balanced growth. Section 4 focuses on the optimal solution for this economy while section 5 finally concludes.

2 Model

2.1 Firms

The production side is based on Arrow (1962) and Romer (1986) where technological progress depends on gross investment. Firms are distributed along the unit interval with a total mass of 1. The production of a representative firm j is given by:

$$y_{t,j} = k_{t,j}^{\alpha}(A_t l_{t,j})^{1-\alpha}, \tag{1}$$

where the technological level shared by all firms is labor augmenting and given by A_t. $k_{t,j}$ and $l_{t,j}$ are respectively the firm specific levels of physical capital and labor at time t. α is the elasticity of physical capital in the production function. We suppose that the aggregate quantity of labor is equal to one:

$$L_t = \int_0^1 l_{t,j} dj = 1. \tag{2}$$

Gross investment is the sum of firms' individuals investments:

$$I_t = \int_0^1 i_{t,j} dj. \tag{3}$$

As in Greiner and Semmler (1996) and Greiner (2003), we assume that the stock of knowledge capital is a by-product of gross investment according to:

$$A_t = \phi \int_{-\infty}^{t} e^{\eta(s-t)} I_s d_s. \tag{4}$$

The introduction of the weighting function $e^{\eta(s-t)}$ implies that new investment contributes more to knowledge than investment further back in time. Differentiating expression (4) with respect to time, we obtain:

$$\dot{A}_t = \phi I_t - \eta A_t, \tag{5}$$

with A_0 given.
$\phi > 0$ shows how much a unit of investment contributes to the increase in knowledge and $\eta > 0$ is the depreciation rate of knowledge capital.
The production function for the overall economy is given by:

$$y_t = f(A_t, k_t) = A_t^{1-\alpha} k_t^{\alpha}. \tag{6}$$

Since the aggregate quantity of labor is equal to one, per capita and aggregate variables are equal in this setting. In the competitive equilibrium, each firm will ignore the impact of its own investment on knowledge accumulation implying that they don't take into account the evolution of knowledge capital. Only a benevolent planner will be aware of the presence of the gross investment externality.

2.2 Preferences

As explained before, our model introduces an endogenous discount rate decreasing in capital. The intertemporal discounted utility function of the representative agent is given by:

$$U(c_t, \{k_t\}_0^t) = \int_0^{\infty} u(c_t) e^{-\theta_t} dt. \tag{7}$$

The discount rate $\theta(t) > 0$ depends on past and current levels of capital in the following way:

$$\theta_t = \int_0^t \rho(k_s) ds, \tag{8}$$

where $\rho(k)$ is the discounting function for which we assume that $\rho'(k) < 0$ such that wealthier agents are more patient than poorer ones.
The utility function is twice continuously differentiable and follows standard

concavity properties, such that $u'(c) > 0, u''(c) < 0$ and $\lim_{c\to 0} u'(c) = \infty$. In the forthcoming analysis, we will use a constant intertemporal elasticity of substitution (CIES) utility function which will take the following functional form:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{9}$$

with $\sigma \geq 0$ representing the inverse of the intertemporal elasticity of substitution.

Before proceeding, we should focus on the importance of the sign of the utility function in endogenous discounting models. In the present case, preferences being recursive, a change in the capital stock will not only have an impact on present utility but also on the way the representative agent perceives future utility gains. In order to compute the marginal utilities, we will rely on the Volterra derivatives with respect to consumption and capital. The Volterra derivative gives the rate at which intertemporal utility changes with respect to a small increase in consumption or capital near a given time t. In the present case, we obtain:

$$U_{tc} = u'(c_t)e^{-\int_0^t \rho(k_s)ds}, \tag{10}$$

$$U_{tk} = -\rho'(k_s)e^{-\int_0^t \rho(k_s)ds} \int_t^{\infty} u(c_s)e^{-\int_t^s \rho(k_\tau)d\tau} ds. \tag{11}$$

It can be observed that the Volterra derivative with respect to consumption is always positive while the one with respect to capital is negative if $u(c) < 0$ and positive if $u(c) > 0$. This suggest that in the case of negative utility a higher capital level decreases intertemporal utility. Subsequently, a higher discount rate can be seen as something good in the sense that it increases total welfare. The lack of economic intuition given by these results suggests that in the present framework, we should use a utility function that only takes positive values as in Schumacher (2011). Note that this is not a more general statement and that the choice of the sign of the utility function should always be based on the respective Volterra derivatives of each model. Given that we will use a a CIES utility function, from now on we will suppose that $\sigma < 1$. This assumption goes against most standard estimates of the intertemporal elasticity of substitution suggesting values below one. Two remarks are in order here: first, most of these estimates have been derived assuming a constant discount rate which is a fundamental difference with the present framework. Second, recent contributions such as Mulligan (2002), Vissing-Jørgensen (2002) as well as Vissing-Jørgensen and Attanasio (2003) provide estimates significantly larger than one. Kapoor and Ravi (2010) as well as Gruber (2013) providing robust estimates in the interval $(2, 3)$.

2.3 Sufficiency conditions and the discount function

If we choose to proceed with a utility function that only takes positive values, a different kind of problem arises: it can be proved that the Mangasarian sufficiency conditions are satisfied only if $u(c) < 0$ (see Schumacher, 2011), so that if one decides to proceed with positive utility, it is necessary to derive appropriate sufficiency conditions. We proceed with our CIES utility function where $\sigma < 1$ and derive these conditions in the following.
Our optimization problem can be written in the following way:

$$\max_{c_t, k_t} \int_0^{\infty} u(c_t) e^{-\int_0^t \rho(k_s) ds} dt, \tag{12}$$

subject to

$$\begin{cases} \dot{k}_t = f(A_t, k_t) - c_t - \delta k_t & \forall t \\ k_t \geq 0, A_t \geq 0, c_t \geq 0 & \forall t \\ \text{with } k_0 \text{ and } A_0 \text{ given,} \end{cases}$$

where $\delta > 0$ is the depreciation rate of physical capital.
As $d\theta_t/dt = \rho(k_t)$, we can write $dt = d\theta_t/\rho(k_t)$ and obtain the following Hamiltonian [1]:

$$\mathcal{H}_\Delta = \frac{c^{1-\sigma} e^{-\theta}}{(1-\sigma)\rho(k)} + \lambda \left(\frac{f(A,k) - c - \delta k}{\rho(k)} \right). \tag{13}$$

The first order condition for the control variable is given by

$$c^{-\sigma} e^{-\theta} = \lambda. \tag{14}$$

Let's rewrite $c = \left(\frac{e^{-\theta}}{\lambda}\right)^{1/\sigma}$ and substitute this in the Hamiltonian in order to obtain the Hamiltonian along the optimal path. Now let's take second order conditions with respect to k to see if the Hamiltonian is indeed concave along the optimal path. In this case, we don't need to do the same for A since we are focusing on the competitive equilibrium. The first-order condition is given by

$$\begin{aligned} \frac{\partial \mathcal{H}_\Delta}{\partial k} &= \lambda \frac{f_k(A,k) - \delta}{\rho(k)} \\ &- \left\{ \left(\frac{e^{-\theta}}{\lambda}\right)^{\frac{1-\sigma}{\sigma}} \frac{e^{-\theta}}{1-\sigma} + \lambda[f(A,k) - \left(\frac{e^{-\theta}}{\lambda}\right)^{1/\sigma} - \delta k] \right\} \frac{\rho'(k)}{\rho(k)^2}, \end{aligned} \tag{15}$$

[1] From now on, we drop the time dependency for convenience.

while the second-order condition is given by

$$\begin{aligned}\frac{\partial^2 \mathcal{H}_\Delta}{\partial k^2} &= -\mathcal{H}_\Delta\left(\frac{\rho''(k)}{\rho(k)} - 2\frac{\rho'(k)^2}{\rho(k)^2}\right) + \lambda\frac{f_{kk}(A,k)}{\rho(k)} \\ &\quad -2\rho'(k)\lambda\left(\frac{f_k(A,k)-\delta}{\rho(k)^2}\right).\end{aligned} \tag{16}$$

Since $\mathcal{H}_\Delta > 0$, the Hamiltonian is concave if the following two conditions are satisfied:

$$\begin{aligned}\rho''(k) &> 2\frac{\rho'(k)^2}{\rho(k)}, \\ f_{kk}(A,k) &< 2\rho'(k)\frac{f_k(A,k)-\delta}{\rho(k)}.\end{aligned}$$

These two conditions imply that the discounting function should be sufficiently convex while the production function should be sufficiently concave. For example, an AK production function would not satisfy the second sufficiency condition.

Given our production function $y = A^{1-\alpha}k^\alpha$, we propose the following discounting function:

$$\rho(k) = \overline{\rho} + \rho_0 e^{-\beta k^\gamma},$$

where $\overline{\rho} \geq 0$. This discounting function is indeed decreasing and convex in capital. Concerning the minimal discount rate $\overline{\rho}$, both a zero or a positive value can be considered. We can imagine that very rich agents can afford not to discount future utilities as well as that individuals will always discount the future for different reasons such as the difficulty to value future enjoyments (see, for example, Becker and Mulligan, 1997), the fact that there is always a relative preference for the present or the possibility of death or extinction (Stern, 2007). We now need to check if our production and discounting functions satisfy our sufficiency conditions. In this specific case, we obtain:

$$\left(\rho_0 + \overline{\rho}e^{\beta k^\gamma}\right)\left(1 + \frac{1-\gamma}{\gamma\beta k^\gamma}\right) > 2\rho_0, \tag{17}$$

$$\frac{\alpha A^{1-\alpha}[(1-\alpha)(\rho_0 + \overline{\rho}e^{\beta k^\gamma}) - 2\beta\gamma\rho_0 k^\gamma]}{k^{1+\gamma-\alpha}} > -2\beta\gamma\rho_0\delta. \tag{18}$$

Concerning the first condition, we can see that the limit of the left hand side of expression (17) when $k \to 0$ is ∞ if $\gamma < 1$ while when $k \to \infty$, the limit is ∞ for any value of γ. We thus choose to impose $\gamma < 1$ in our discounting

function such that the condition is satisfied for low and high values of the capital stock.
Concerning condition (18), when $k \to 0$, the limit of the left hand side is ∞. We also impose that $\gamma > \alpha$ so that when $k \to \infty$, the limit is ∞ as well. The second condition is then satisfied for low and high values of the capital stock provided that $\alpha < \gamma < 1$. With this restriction in hand, we can proceed with the competitive equilibrium.

2.4 Competitive equilibrium

Since we are in the competitive case, the physical capital stock evolves according to:

$$\dot{k} = rk + w - c - \delta k \tag{19}$$

where r is the interest rate and w is the wage. The interest rate and the wage are given by marginal productivities:

$$r = \alpha A^{1-\alpha} k^{\alpha-1}, \tag{20}$$
$$w = (1-\alpha) A^{1-\alpha} k^{\alpha}. \tag{21}$$

By taking the derivative of expression (8) with respect to time we obtain:

$$\dot{\theta} = \overline{\rho} + \rho_0 e^{-\beta k^{\gamma}}, \tag{22}$$

which can be used as a second state variable concerning the evolution of the discount rate. The representative agent then solves the following problem:

$$\max_{c,k} \int_0^{\infty} \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} dt, \tag{23}$$

subject to

$$\begin{cases} \dot{k} = (r-\delta)k + w - c & \forall t \\ \dot{\theta} = \overline{\rho} + \rho_0 e^{-\beta k^{\gamma}} & \forall t \\ k \geq 0, c \geq 0 & \forall t \\ \text{with } k_0 \text{ given.} & \end{cases}$$

We write the present value Hamiltonian of the above system:

$$\mathcal{H} = \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} + \lambda[(r-\delta)k + w - c] - \mu(\overline{\rho} + \rho_0 e^{-\beta k^{\gamma}}). \tag{24}$$

The first order necessary and sufficient conditions for optimality are:

$$\begin{aligned} c^{-\sigma}e^{-\theta} &= \lambda, && (25) \\ \lambda(r-\delta)+\beta\mu\gamma k^{\gamma-1}\rho_0 e^{-\beta k^\gamma} &= -\dot{\lambda}, && (26) \\ -\frac{c^{1-\sigma}}{1-\sigma}e^{-\theta} &= \dot{\mu}, && (27) \\ \lim_{t\to\infty}\mathcal{H}_t &= 0. && (28) \end{aligned}$$

As explained in Michel (1982), the transversality condition is modified in infinite horizon problems and the appropriate one is given by expression (28).

By solving the model, using the equilibrium values for r and w and adding the evolution of knowledge capital we obtain the following dynamical system:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{\alpha A^{1-\alpha}k^{\alpha-1}-\delta-\overline{\rho}-\rho_0 e^{-\beta k^\gamma}}{\sigma}+\frac{\mu\gamma\beta c^{\sigma}k^{\gamma-1}\rho_0 e^{-\beta k^\gamma}}{\sigma e^{-\theta}}, && (29) \\ \dot{k} &= A^{1-\alpha}k^{\alpha}-c-\delta k, && (30) \\ \dot{\mu} &= -\frac{c^{1-\sigma}}{1-\sigma}e^{-\theta}, && (31) \\ \dot{A} &= \phi(A^{1-\alpha}k^{\alpha}-c)-\eta A, && (32) \\ \lim_{t\to\infty}\mathcal{H}_t &= 0, && (33) \end{aligned}$$

with k_0 and A_0 given.

Lemma 1:
The previous dynamical system can be reduced to the following three dimensional system:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{\alpha A^{1-\alpha}k^{\alpha-1}-\delta-\overline{\rho}-\rho_0 e^{-\beta k^\gamma}}{\sigma} && (34) \\ &\quad +\frac{\beta\gamma\rho_0 k^{\gamma-1}}{\sigma(\rho_0+\overline{\rho}e^{\beta k^\gamma})}\left(\frac{c}{1-\sigma}+\dot{k}\right), \\ \frac{\dot{k}}{k} &= A^{1-\alpha}k^{\alpha-1}-\frac{c}{k}-\delta, && (35) \\ \frac{\dot{A}}{A} &= \phi(A^{-\alpha}k^{\alpha}-\frac{c}{A})-\eta, && (36) \end{aligned}$$

with k_0 and A_0 given.

Proof. First note that the Hamiltonian is autonomous. Differentiating the Hamiltonian with respect to time we obtain:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial c}\dot{c} + \frac{\partial \mathcal{H}}{\partial k}\dot{k} + \frac{\partial \mathcal{H}}{\partial \theta}\dot{\theta} + \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda} - \frac{\partial \mathcal{H}}{\partial \mu}\dot{\mu}. \tag{37}$$

Using the first order conditions for optimality:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} - \dot{\lambda}\dot{k} + \dot{k}\dot{\lambda} + \dot{\mu}\dot{\theta} - \dot{\theta}\dot{\mu}, \tag{38}$$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}. \tag{39}$$

Since the Hamiltonian is autonomous, $\frac{\partial \mathcal{H}}{\partial t} = 0$. Combining this result with the transversality condition, $\lim_{t\to\infty} \mathcal{H}_t = 0$, implies that the Hamiltonian takes the value zero along the optimal trajectory.
We now have a solution for μ by transforming the Hamiltonian:

$$\mu = \frac{c^{-\sigma}e^{-\theta}[c + (1-\sigma)\dot{k}]}{(1-\sigma)(\overline{\rho} + \rho_0 e^{-\beta k^{\gamma}})}. \tag{40}$$

Replacing μ by its value in the differential equation for consumption yields the desired result. □

Expression (34), (35) and (36) capture the dynamics of our economy. The consumption equation can be separated into two terms: the first one is similar to a neoclassical accumulation equation where the depreciation rate of physical capital and the endogenous discount rate are subtracted from the marginal productivity of capital. The second is a positive term taking into account the indirect effect of a decreasing discount rate on the growth rate of consumption. Given this additional term, the growth rate of consumption is an increasing function of the consumption level and of the growth rate of capital.
Concerning the physical and knowledge capital accumulation equations, both processes are similar. In fact, if we set $\phi = 1$ and $\delta = \eta$, both variables can be merged into a single one similarly to Romer (1986). We are going to assume in the following that the depreciation rate of knowledge capital is higher than the one for physical capital so that $\delta < \eta$. Several empirical studies have confirmed that this is indeed the case in actual economies (see, for example, Nadiri and Prucha, 1996; Park et al., 2006). The reason behind these results might be that since technological change has been accelerating in western countries (at least before the 2008 crisis), it was to be expected that existing knowledge would become more rapidly obsolete.

3 The dynamics

In this section, we study the different dynamic behaviors that can be observed in the competitive case. We first focus on the existence of a steady-state equilibrium and then analyse balanced-growth.

3.1 Steady-state equilibrium

Definition 1:
A steady-state equilibrium of this economy is a solution (c_t, k_t, A_t) to equations (34), (35) and (36) given k_0 and A_0, such that $g_c = g_k = g_A = 0$, where g represents the growth rate of the respective endogenous variables.

Setting $\dot{c} = \dot{k} = \dot{A} = 0$, we obtain:

$$\overline{c} = \frac{1}{\gamma\beta\rho_0}\left[\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho} - \rho_0 e^{-\beta\overline{k}^\gamma}\right](\sigma-1)(\rho_0 + \overline{\rho}e^{\beta\overline{k}^\gamma})\overline{k}^{1-\gamma}, \tag{41}$$

$$\overline{c} = \left[\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta\right]\overline{k}, \tag{42}$$

$$\overline{A} = \frac{\phi\delta}{\eta}\overline{k}. \tag{43}$$

In this case, we can proceed with the first two steady-state equations (41) and (42) since at the steady-state A is a function of k only. We now make a crucial assumption concerning some parameters of the model. We suppose that $\delta + \overline{\rho} < \alpha(\phi\delta/\eta)^{1-\alpha} < \delta + \overline{\rho} + \rho_0$ so that at a steady-state equilibrium, accumulation does not proceed only because of the endogenous discounting effect and not because of a too low rate of interest compared with the depreciation and the minimal discount rates.

Proposition 1: In this economy:

(i) A necessary and sufficient condition for the existence of a steady-state is that the steady-state capital stock satisfies the following condition:

$$\overline{k} < \left\{-\frac{1}{\beta}\ln\left[\frac{\alpha(\phi\delta/\eta)^{1-\alpha} - \delta - \overline{\rho}}{\rho_0}\right]\right\}^{1/\gamma}. \tag{44}$$

(ii) There are two steady-state equilibria: the trivial one $(\overline{c}_0, \overline{k}_0) = (0, 0)$ and $(\overline{c}_1, \overline{k}_1) > (0, 0)$.

(iii) $(\overline{c}_0, \overline{k}_0)$ is unstable while $(\overline{c}_1, \overline{k}_1) \neq (0,0)$ is saddle-path stable.

Proof. We start with part (i) of the proposition. As can be seen from expression (41), the existence of a steady-state equilibrium with values for $\overline{c} \geq 0$ is possible if and only if

$$\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho} < \rho_0 e^{-\beta\overline{k}^\gamma}, \tag{45}$$

given that $\sigma < 1$ by assumption. Rearranging the expression as a condition on the steady-state level of capital, we obtain expression (44).
Concerning part (ii), we study the following function:

$$\begin{aligned} g(k) &= \frac{1}{\gamma\beta\rho_0}\left[\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}\right](\sigma-1)(\rho_0 + \overline{\rho}e^{\beta k^\gamma})k^{1-\gamma} \\ &\quad - \left[\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta\right]k, \end{aligned} \tag{46}$$

which is continuous for all $k \in (0, +\infty)$. A steady-state equilibrium is given by values of $\overline{k}$ for which $g(\overline{k}) = 0$. We first take the limits of our function:

$$\lim_{k\to 0} g(k) = 0, \lim_{k\to+\infty} g(k) = -\infty.$$

Its derivative is given by

$$\begin{aligned} g'(k) &= \frac{1-\gamma}{\gamma\beta\rho_0}\left[\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}\right](\sigma-1)(\rho_0 + \overline{\rho}e^{\beta k^\gamma})k^{-\gamma} \\ &\quad + \frac{1}{\rho_0}\left[\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}\right](\sigma-1)\overline{\rho}e^{\beta k^\gamma} \\ &\quad + (\sigma-1)(\overline{\rho} + \rho_0 e^{-}\beta k^\gamma) - \left[\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta\right]. \end{aligned} \tag{47}$$

And the limits of the derivative are

$$\lim_{k\to 0} g'(k) = +\infty, \lim_{k\to+\infty} g'(k) = -\infty.$$

These results indicate to us that there are at least two steady-states and always an even number of them. The first one displays consumption and capital levels equal to zero.
We want to prove next that we only have two steady-states. In order to do

so we show that $g'(k) = 0$ has only one solution so that there is a unique inflexion point. We set $g'(k) = 0$ and study this function:

$$\frac{1}{\gamma\beta\rho_0}\left[\alpha\left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta - \overline{\rho}\right](\sigma-1)[(1-\gamma)(\rho_0 + \overline{\rho}e^{\beta k^\gamma}k^{-\gamma} + \beta\gamma\overline{\rho}e^{\beta k^\gamma}]$$
$$+(1-\gamma)\rho_0[(1-\gamma)(\overline{\rho} + \rho_0 e^{-\beta k^\gamma})k^{-\gamma} + \beta\gamma\overline{\rho}]$$
$$= (1-\sigma)(\overline{\rho} + \rho_0 e^{-\beta k^\gamma}) + \left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta. \tag{48}$$

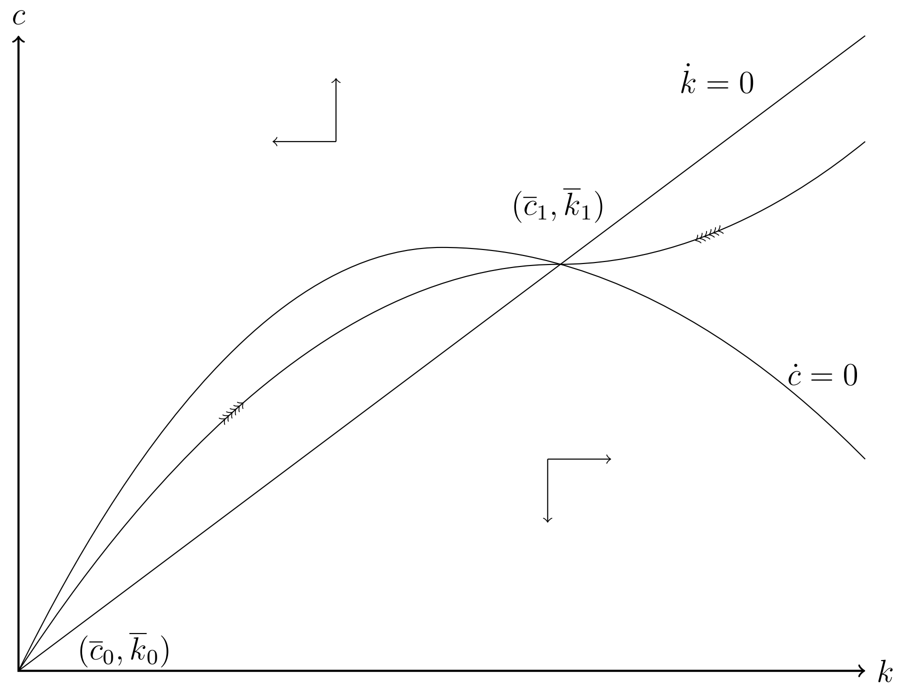
We label the left hand side $m(k)$ and the right hand side $n(k)$. the limits are given by

$$\lim_{k\to 0} m(k) = +\infty, \lim_{k\to+\infty} m(k) = -\infty,$$
$$\lim_{k\to 0} n(k) = (1-\sigma)(\rho_0 + \overline{\rho}) + \left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta > 0,$$
$$\lim_{k\to+\infty} n(k) = (1-\sigma)\overline{\rho} + \left(\frac{\phi\delta}{\eta}\right)^{1-\alpha} - \delta > 0. \tag{49}$$

We can moreover observe that $m(k)$ is necessarily decreasing for a sufficiently high level of k while $n(k)$ is decreasing in k and becomes constant in the limit. Using these results we can conclude that $m(k)$ and $n(k)$ only cross once implying that there is a unique inflexion point and thus two steady-states: $(\overline{c}_0, \overline{k}_0) = (0, 0)$ and $(\overline{c}_1, \overline{k}_1) > (0, 0)$.
For part (iii), we're going to proceed with a phase diagram analysis. The steady-state curves are given by expressions (41) and (42). Expression (42) is increasing and linear and goes from $(c, k) = (0, 0)$ to $(c, k) = (+\infty, +\infty)$. Expression (41) goes from $(c, k) = (0, 0)$ to $(c, k) = (-\infty, +\infty)$. Since we know that there are two steady-states, one of which does not exhibit positive levels of capital and consumption, we can draw the phase diagram given in figure 1. As can be seen, $(\overline{c}_0, \overline{k}_0) = (0, 0)$ is unstable while $(\overline{c}_1, \overline{k}_1) > (0, 0)$ is saddle-path stable. □

The existence of a saddle-path stable steady-state implies that economies starting with an initial level of capital that is lower than a certain level will not be able to reach growth in the long run. This is the case even if the net rate of interest is positive and higher than the minimal discount rate. Moreover, the steady-state capital stock is bounded above by condition (44) implying that a steady-state is only compatible with relatively low values of the capital stock. Since long run growth is a possible outcome of the model given the learning by doing process, this steady-state can be interpreted as a poverty trap (Azariadis, 1996; Azariadis and Stachurski, 2005). The reason for this outcome is that a relatively small capital stock implies in return a large discount rate and thus low incentives concerning capital accumulation. The relatively low investment affecting as well the accumulation of knowledge capital. The overconsumption effect due to endogenous discounting makes the dynamic path unsustainable in the long run driving the economy toward a steady-state of stagnation.

Figure 1: Phase diagram



The phase diagram exposed in Figure 1 includes both steady-states curves ($\dot{k} = 0$ and $\dot{c} = 0$) as well as the stable arm given by the curve with arrows. Given the arrows of motion, the first steady state $(\overline{c}_0, \overline{k}_0) = (0, 0)$ is unstable while the second $(\overline{c}_1, \overline{k}_1) > (0, 0)$ is saddle-path stable.

3.2 Asymptotic balanced growth path equilibrium

In this section, we study the existence of balanced growth where consumption, output and capital grow without bound at a common positive constant rate. As it has been proved in Palivos et al. (1997), a necessary and sufficient condition for the existence of a BGP is that the elasticity of marginal utility and the discount rate be constant along this path. This is not the case in our model since the discount rate is endogenous. Nevertheless, we can apply the concept of an asymptotically balanced growth path (ABGP) to our problem.

Definition 2:
An ABGP equilibrium of this economy is a solution (c_t, k_t, A_t) to equations (34), (35) and (36) given k_0 and A_0, such that $\lim_{c\to\infty} g_c = \lim_{k\to\infty} g_k = \lim_{A\to\infty} g_A > 0$.

In order to draw some conclusions from the ABGP, the usual method is to operate on variables which are constant along the ABGP. From our dynamical system, we know that a constant growth rate for capital obtains if $z = k/A$ and $x = c/A$ are constant. x is the consumption to knowledge capital ratio while z is the capital to knowledge capital ratio.
We first determine the differential equations for z and x:

$$g_z = -\delta + \eta - \frac{x}{z} + \phi x + (1-\phi z)z^{\alpha-1}, \tag{50}$$

$$\begin{aligned} g_x &= \frac{\alpha z^{\alpha-1} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} + \eta + \phi x - \phi z^\alpha \\ &\quad + \frac{[(1-\sigma)(z^{\alpha-1}-\delta) + \sigma x]\gamma\rho_0\beta k^\gamma}{\sigma(1-\sigma)(\rho_0 + \overline{\rho}e^{\beta k^\gamma})}, \end{aligned} \tag{51}$$

with $z(0)$ given.

Proposition 2: In this economy:

(i) An ABGP equilibrium exists only if $\overline{\rho} > 0$.

(ii) Since $\delta < \eta$ and $\sigma < 1$, there are two ABGP equilibria.

(iii) The ABGP giving the higher growth rate is saddle-path stable while the one yielding the lower growth rate cannot be a saddle-point.

(iv) The ABGP giving the lower growth rate can be locally indeterminate, unstable or generate a possible limit-cycle through a Hopf bifurcation.

Proof. We will first show that $\overline{\rho} > 0$ is a necessary condition for the existence of an ABGP equilibrium. Let's first suppose that $\overline{\rho} = 0$. In this case, the limit of the differential equation (51) is given by

$$\begin{aligned} \lim_{k\to\infty} g_x &= \lim_{k\to\infty}\left(\frac{\alpha z^{\alpha-1} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}}{\sigma} + \eta + \phi x - \phi z^\alpha\right) \\ &\quad + \lim_{k\to\infty}\left\{\frac{[(1-\sigma)(z^{\alpha-1}-\delta) + \sigma x]\gamma\beta k^\gamma}{\sigma(1-\sigma)}\right\}, \\ &= +\infty, \end{aligned} \tag{52}$$

since $\sigma < 1$. An ABGP equilibrium cannot existence in this case since the consumption to knowledge capital ratio grows constantly. Now if $\overline{\rho} > 0$, the limit of expression (51) is indeterminate since both k^γ and $e^{\beta k^\gamma}$ grow continuously. Making use of L'Hopital's rule:

$$\lim_{k\to\infty} \frac{k^\gamma}{e^{\beta k^\gamma}} = \lim_{k\to\infty} \frac{1}{\beta e^{\beta k^\gamma}} = 0. \tag{53}$$

We can conclude that the last term on the right hand side converges to zero and thus $\lim_{t\to\infty} \dot{x}$ exists and is finite.
From expressions (50) and (51), we obtain our steady-states equations:

$$\begin{aligned}
\lim_{k\to\infty} g_z &= -\delta + \eta - \lim_{k\to\infty} \frac{x}{z} + \phi \lim_{k\to\infty} x + \lim_{k\to\infty} (1-\phi z) z^{\alpha-1}, &(54)\\
\lim_{k\to\infty} g_x &= \frac{\alpha \lim_{k\to\infty} z^{\alpha-1} - \delta - \overline{\rho}}{\sigma} + \eta + \phi \lim_{k\to\infty} x - \phi \lim_{k\to\infty} z^\alpha, &(55)\\
&= 0.
\end{aligned}$$

We can now study the possible existence of ABGP equilibria. In order to do so we first label $\lim_{k\to\infty} z = Z$ and $\lim_{k\to\infty} x = X$. Notice first that our assumption $\delta < \eta$ requires $\phi Z > 1$ for $A^{1-\alpha}k^\alpha > c$. From expression (54), we get $\overline{X}$ that we plug into (55) to obtain:

$$h(Z) = \eta + \frac{(\eta - \delta)\phi Z}{1 - \phi Z} + \frac{\alpha Z^{\alpha-1} - \delta - \overline{\rho}}{\sigma} \tag{56}$$

A Z that solves $h(Z) = 0$ also solves $h_1(Z) = h(Z)Z^{1-\alpha}(1 - \phi Z) = 0$ with $Z^{1-\alpha}(1 - \phi Z) \neq 0$. We choose to analyse the function $h_1(Z)$ instead of $h(Z)$ where:

$$h_1(Z) = \frac{\alpha(1-\phi Z)}{\sigma} + \left(\eta - \frac{\overline{\rho} + \delta}{\sigma}\right) Z^{1-\alpha} + \phi \left(\frac{\overline{\rho} + \delta}{\sigma} - \delta\right) Z^{2-\alpha}, \tag{57}$$

which is continuous for all $Z \in (0, +\infty)$. The derivative of $h_1(Z)$ is given by

$$\begin{aligned}
h_1'(Z) &= -\frac{\alpha\phi}{\sigma} + (1-\alpha)\left(\eta - \frac{\overline{\rho}+\delta}{\sigma}\right) Z^{-\alpha} \\
&\quad + \phi(2-\alpha)\left(\frac{\overline{\rho}+\delta}{\sigma} - \delta\right) Z^{1-\alpha}. &(58)
\end{aligned}$$

Concerning the terms in $h_1(Z)$, it should be notice that the last term on the right hand side takes positive values since $\sigma < 1 + \overline{\rho}/\delta$ given our previous discussion about endogenous discounting models. As stated before, $\delta < \eta$ implies $\phi Z > 1$ so that the first term takes negative values. We thus only

need to explore two cases which exhibit different signs concerning the second term of expression (57):
Case 1: $\eta \leq (\overline{\rho} + \delta)/\sigma$
We first take the limits of $h_1(Z)$ and $h_1'(Z)$:

$$\lim_{Z \to 0} h_1(Z) = \alpha/\sigma > 0, \lim_{Z \to +\infty} h_1(Z) = +\infty,$$
$$\lim_{Z \to 0} h_1'(Z) = -\infty, \lim_{Z \to +\infty} h_1'(Z) = +\infty.$$

These results combined with the fact that $h_1''(Z) > 0$ for all $Z > 0$ imply that in this case we have two ABGP equilibria.
Case 2: $\eta > (\overline{\rho} + \delta)/\sigma$
We take the limits of $h_1(Z)$ and $h_1'(Z)$ in this case:

$$\lim_{Z \to 0} h_1(Z) = \alpha/\sigma > 0, \lim_{Z \to +\infty} h_1(Z) = +\infty,$$
$$\lim_{Z \to 0} h_1'(Z) = +\infty, \lim_{Z \to +\infty} h_1'(Z) = +\infty.$$

Since $h_1''(Z) = 0$ has only one solution given by $Z = \alpha(\overline{\rho} + \delta - \eta\sigma)/\phi(2 - \alpha)[\overline{\rho} + (1 - \sigma)\delta] > 0$, only two ABGP equilibria can exist in this case as well. We can now study the local stability of the system around the asymptotic steady-state. In order to realize this task, we first compute the Jacobian matrix at (X, Z) using expressions (55) and (54) to obtain:

$$J = \begin{bmatrix} \phi X & -\left[\frac{\alpha(1-\alpha)}{\sigma} Z^{\alpha-2} + \phi\alpha Z^{\alpha-1}\right] X \\ -(1 - \phi Z) & X/Z - (1 - \alpha)Z^{\alpha-1} - \phi\alpha Z^{\alpha} \end{bmatrix}. \tag{59}$$

The Determinant of our matrix is given by

$$Det(J) = (1 - \phi Z)X \left[\frac{\phi(\eta - \delta)}{(1 - \phi Z)^2} - \frac{\alpha(1 - \alpha)}{\sigma} Z^{\alpha-2}\right]. \tag{60}$$

We can further notice that

$$h'(Z) = \frac{\phi(\eta - \delta)}{(1 - \phi Z)^2} - \frac{\alpha(1 - \alpha)}{\sigma} Z^{\alpha-2}. \tag{61}$$

A necessary and sufficient condition for saddle-path stability is $Det(J) < 0$. We thus need to study the sign of $h'(Z)$ at both ABGP equilibria. We already know that $Z \in (\phi^{-1}, +\infty)$ and the function $h(Z)$ has the following properties:

$$\lim_{Z \to \phi^{-1}} h(Z) = -\infty, \lim_{Z \to +\infty} h(Z) = \delta - \frac{(\delta + \overline{\rho})}{\sigma} < 0,$$
$$\lim_{Z \to \phi^{-1}} h'(Z) = -\infty, \lim_{Z \to +\infty} h'(Z) = 0.$$

From the previous results, we know that $1 - \phi Z < 0$, $h'(Z) > 0$ at the first (lower) Z and $h'(Z) < 0$ at the second (larger) Z. Consequently, the first ABGP (lower Z) is saddle-path stable while the second one (higher Z) cannot be a saddle point.
The growth rate of consumption along an ABGP is given by

$$\lim_{k \to +\infty} g_c = \frac{\alpha Z^{\alpha-1} - \delta - \overline{\rho}}{\sigma}. \tag{62}$$

The ABGP with the lower Z which is saddle-path stable also exhibits the highest growth rate since $\alpha < 1$.
We can now study the possible behavior of the ABGP giving the lower growth rate. From our previous result, we know that in this case $Det(J) > 0$. The stability outcome will thus depend on the Trace of the Jacobian matrix which is given by

$$Tr(J) = \frac{(\phi Z + 1)(\eta - \delta)}{1 - \phi Z} + (1 - \alpha)\phi Z^{\alpha} + \alpha Z^{\alpha-1}. \tag{63}$$

If $Tr(J) > 0$ the ABGP is unstable while if $Tr(J) < 0$ it is locally indeterminate. Finally, if $Tr(J) = 0$ the ABGP undergoes a Hopf bifurcation which can generate a possible limit-cycle. In this case, the system would permanently oscillate around the ABGP equilibrium. □

The existence of an ABGP thus requires a positive minimal discount rate. Indeed, without this condition, the growth rate of consumption would divert toward infinity. The model then requires that at every period of time, the representative agent keeps a preference for the present. As expressed before, this does not seem to be a too strong requirement. The second condition is more important and tells us that since the elasticity of marginal utility is lower than one in this framework, we obtain the existence of two balanced growth paths and thus global indeterminacy. This result comes directly from the restriction on our utility function due to endogenous discounting. Concerning the behavior of actual economies, this result gives us a potential explanation concerning the fact that similar developed economies can display differences in terms of growth rates in the long run (Acemoglu, 2008). The ABGP with the higher growth rate is saddle-path stable and in the case where the second ABGP is unstable, this is the only long run equilibrium to which all economies will converge. A more interesting case is obtained when the second ABGP is locally indeterminate, since in this case, some economies will converge toward the ABGP with lower growth. However, these economies will experience differences in terms of growth rates during the transition to the ABGP giving a potential explanation to short run differences among similar economies. Finally, if the ABGP undergoes a Hopf bifurcation, it is possible that these economies experience everlasting fluctuations around the low growth ABGP equilibrium. A last interesting result is that the joint existence of a steady-state of stagnation if the initial capital stock is too low and of balanced growth is compatible with the so called twin peaks of economic growth. As documented empirically (Quah, 1996; Jones, 1997; Beaudry et al., 2005), the world distribution of income has been changing from a somewhat unimodal distribution toward a bimodal one with a group of countries experiencing continuous growth while another one seems to be stuck at lower levels of development and is thus unable to catch-up.

4 Optimal solution

The only difference between the competitive and the optimal case is that the social planner is aware of the evolution of knowledge capital. The feasibility constraint is given by

$$\dot{k} = A^{1-\alpha}k^{\alpha} - c - \delta k. \tag{64}$$

The social planner solves the following problem:

$$\max_{c,k,A} \int_0^{\infty} \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} dt, \tag{65}$$

subject to

$$\begin{cases} \dot{k} = A^{1-\alpha}k^{\alpha} - c - \delta k & \forall t \\ \dot{A} = \phi(A^{1-\alpha}k^{\alpha} - c) - \eta A & \forall t \\ \dot{\theta} = \overline{\rho} + \rho_0 e^{-\beta k^{\gamma}} & \forall t \\ k \geq 0, A \geq 0, c \geq 0 & \forall t \\ \text{with } k_0 \text{ and } A_0 \text{ given.} \end{cases}$$

The introduction of the additional constraint does not change the results concerning second-order conditions and we can proceed with the first order ones which are also sufficient. We have the following present value Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \frac{c^{1-\sigma}}{1-\sigma} e^{-\theta} + \lambda_1 (A^{1-\alpha}k^{\alpha} - c - \delta k) + \lambda_2 [\phi(A^{1-\alpha}k^{\alpha} - c) - \eta A] \\ &\quad - \mu_1 (\overline{\rho} + \rho_0 e^{-\beta k^{\gamma}}). \end{aligned} \tag{66}$$

The first order necessary and sufficient conditions for optimality are

$$c^{-\sigma} e^{-\theta} = \lambda_1 + \phi \lambda_2, \tag{67}$$

$$
\begin{aligned}
-\dot{\lambda}_1 &= (\lambda_1 + \phi\lambda_2)\alpha A^{1-\alpha}k^{\alpha-1} - \lambda_1\delta + \mu_1\beta\gamma k^{\gamma-1}\rho_0 e^{-\beta k^\gamma}, && (68)\\
-\dot{\lambda}_2 &= (\lambda_1 + \phi\lambda_2)(1-\alpha)A^{-\alpha}k^{\alpha} - \lambda_2\eta, && (69)\\
-\dot{\mu} &= \frac{c^{1-\sigma}}{1-\sigma}e^{-\theta}, && (70)\\
\lim_{t\to\infty}\mathcal{H}_t &= 0. && (71)
\end{aligned}
$$

By solving the model and using Lemma 1, we obtain the following dynamical system:

$$
\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\sigma}\left[\alpha A^{1-\alpha}k^{\alpha-1} + \phi(1-\alpha)A^{-\alpha}k^{\alpha} - \delta - \overline{\rho} - \rho_0 e^{-\beta k^\gamma}\right]\\
&\quad + \frac{\beta\gamma\rho_0 k^{\gamma-1}}{\sigma(\rho_0 + \overline{\rho}e^{\beta k^\gamma})}\left[\frac{c}{1-\sigma} + \dot{k} - \frac{\lambda_2}{\lambda_1+\phi\lambda_2}(\eta - \phi\delta)\right]\\
&\quad - \frac{\phi\lambda_2}{\sigma(\lambda_1+\phi\lambda_2)}(\eta-\delta), && (72)\\
-\frac{\dot{\lambda}_1}{\lambda_1} &= \frac{\lambda_1+\phi\lambda_2}{\lambda_1}\alpha A^{1-\alpha}k^{\alpha-1} - \delta\\
&\quad + \frac{\beta\gamma\rho_0 k^{\gamma-1}}{(\rho_0 + \overline{\rho}e^{\beta k^\gamma})}\left[\frac{\lambda_1+\phi\lambda_2}{\lambda_1}\left(\frac{c}{1-\sigma} + \dot{k}\right) - \frac{\lambda_2}{\lambda_1}(\eta-\phi\delta)\right], && (73)\\
-\frac{\dot{\lambda}_2}{\lambda_2} &= \frac{\lambda_1+\phi\lambda_2}{\lambda_2}(1-\alpha)A^{-\alpha}k^{\alpha} - \eta && (74)\\
\frac{\dot{k}}{k} &= A^{1-\alpha}k^{\alpha-1} - \frac{c}{k} - \delta, && (75)\\
\frac{\dot{A}}{A} &= \phi(A^{-\alpha}k^{\alpha} - \frac{c}{A}) - \eta, && (76)
\end{aligned}
$$

with k_0 and A_0 given.

The main difference between the competitive and the optimal outcome is that the marginal cost of a unit of consumption is higher in the latter since the planner takes into account the positive impact that investment has on knowledge accumulation. As in Romer (1986), the policy consists in giving incentives to increase investment: this can be done for example by introducing a lump-sum tax which is used to subsidize investment.

We are now going to study the existence of asymptotic balanced growth in the optimal case. As before, we can apply the concept of an asymptotic balanced growth path equilibrium to our economy since the discount rate is only constant in the limit. As can be noticed from our dynamical system in this case, in addition to $z = k/A$ and $x = c/A$, $\omega = \lambda_1/\lambda_2$ should also be constant in order to obtain a constant and equal growth rate for capital, consumption and knowledge. We can now determine the differential equations for x, z and ω:

$$
\begin{aligned}
g_z &= -\delta+\eta-\frac{x}{z}+\phi x+(1-\phi z)z^{\alpha-1}, & (77)\\
g_x &= \frac{\alpha z^{\alpha-1}+\phi(1-\alpha)z^{\alpha}-\delta-\overline{\rho}-\rho_0 e^{-\beta k^{\gamma}}-\phi(\eta-\delta)/(\omega+\phi)}{\sigma}\\
&\quad +\frac{\beta\gamma\rho_0 k^{\gamma}}{\sigma(\rho_0+\overline{\rho}e^{\beta k^{\gamma}})}\left[\frac{x/z}{1-\sigma}+g_k-\frac{\eta-\phi\delta}{(\omega+\phi)k}\right]+\eta+\phi x-\phi z^{\alpha}, & (78)\\
g_\omega &= -(1+\frac{\phi}{\omega})\alpha z^{\alpha-1}+\delta+(\omega+\phi)(1-\alpha)z^{\alpha}-\eta\\
&\quad -\frac{(1+\phi/\omega)\beta\gamma\rho_0 k^{\gamma}}{(\rho_0+\overline{\rho}e^{\beta k^{\gamma}})}\left[\frac{x/z}{1-\sigma}+g_k-\frac{\eta-\phi\delta}{(\omega+\phi)k}\right], & (79)
\end{aligned}
$$

with z_0 given.

Proposition 3: In the optimal economy:

(i) An ABGP equilibrium exists only if $\overline{\rho}>0$.

(ii) The ABGP equilibrium is unique.

Proof. Concerning (i), the proof is similar to the first part of Proposition 2. If $\overline{\rho}=0$, g_x and g_ω converge respectively to $+\infty$ and $-\infty$ which is not compatible with the existence of an ABGP. If $\overline{\rho}>0$, the limit in both cases exists and is finite since $\lim_{k\to\infty} k^{\gamma}/e^{\beta k^{\gamma}}=0$.
Concerning (ii), labelling $\lim_{k\to\infty} z=Z$, $\lim_{k\to\infty} x=X$ and $\lim_{k\to\infty}\omega=\Omega$, we obtain our steady-state equations:

$$
\begin{aligned}
\lim_{k\to\infty} g_z &= -\delta+\eta-\frac{X}{Z}+\phi X+(1-\phi Z)Z^{\alpha-1}, & (80)\\
\lim_{k\to\infty} g_x &= \frac{\alpha Z^{\alpha-1}+\phi(1-\alpha)Z^{\alpha}-\delta-\overline{\rho}-\phi(\eta-\delta)/(\phi+\Omega)}{\sigma}\\
&\quad +\eta+\phi X-\phi Z^{\alpha}, & (81)\\
\lim_{k\to\infty} g_\omega &= -(1+\frac{\phi}{\Omega})\alpha Z^{\alpha-1}+\delta+(\phi+\Omega)(1-\alpha)Z^{\alpha}-\eta, & (82)\\
&= 0.
\end{aligned}
$$

From expressions (80) and (82), we obtain respectively values for X and $\phi+\Omega$ that we plug into expression (81) in order to obtain

$$
\Omega=\frac{\phi\alpha}{-\alpha+(\delta+\overline{\rho}-\sigma\eta)Z^{1-\alpha}-\frac{\sigma\phi(\eta-\delta)}{1-\phi Z}Z^{2-\alpha}}. \tag{83}
$$

Ω is now a function of Z and of parameters only. The limits of Ω are given by

$$\lim_{Z\to 0} \Omega = -\phi, \lim_{Z\to +\infty} \Omega = 0.$$

We can now use these results in order to study the existence and uniqueness of the ABGP. We study the following function:

$$d(Z) = -(1 + \frac{\phi}{\Omega(Z)})\alpha Z^{\alpha-1} + \delta + (\phi + \Omega(Z))(1-\alpha)Z^{\alpha} - \eta. \tag{84}$$

We have the following limits:

$$\begin{aligned}\lim_{Z\to 0} d(Z) &= \eta(\sigma - 1) - \overline{\rho} < 0, \\ \lim_{Z\to \infty} d(Z) &= +\infty.\end{aligned}$$

Moreover, it can be proved that $d(Z)$ is increasing so that there is a unique value of Z for which $d(Z) = 0$ and thus a unique ABGP. □

The fact that the ABGP is unique implies that an appropriate policy might play an important role in two ways: it will first allow economies stuck in the poverty trap to overcome the latter and converge toward balanced growth. Second, it will eliminate the growth differential observed between the two balanced growth paths of the competitive case and bring these economies to a ABGP with higher growth. As explained before, the policy should consist in giving incentives to increase investment. It should be noticed that this will induce two effects: the first being the standard increase in the interest rate while the second being a decrease in the discount rate due to an increase in the capital stock. This latter effect will further enhance capital accumulation.

5 Conclusion

The present paper has focused on the interaction between endogenous growth driven by a learning by doing process and endogenous discounting. The necessity to consider a utility function that takes only positive values due to our endogenous discounting formulation has important consequences in this framework. We proved the existence of a saddle-path stable steady-state which can be interpreted as a poverty trap. This steady-state exists despite the fact that the net rate of interest is higher then the minimal discount rate at the steady-state. This result might partially explain why countries that have some growth potential are stuck at lower levels of development and seem unable to catch-up. In the case of balanced growth, a positive utility function combined with a rate of depreciation higher for knowledge capital implies the existence of two asymptotic balanced growth paths. This global indeterminacy result is a potential explanation concerning the growth differential observed in the long run among otherwise similar developed economies. Moreover, the dynamic behavior of the low growth ABGP equilibrium gives us a potential explanation for short run growth differences (the equilibrium can be locally indeterminate) or long run fluctuations (potential existence of a limit-cycle). The joint existence of a poverty trap and balanced growth depending on the initial level of capital is in accordance with the so-called twin peaks of economic growth. The results concerning the optimal outcome show that the latter is globally determinate implying that an appropriate policy might allow all economies to converge to the same asymptotic balanced growth path in the long run.

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