A fixed point theorem and equilibria of abstract economies with $w$-upper semicontinuous correspondences.

By Monica Patriche and Carlos Hervés Beloso
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Abstract

We introduce the notion of \(w\)-upper semicontinuous correspondences and give a new fixed-point theorem. We also introduce the notion of correspondences with \(e\)-USS-property. These results can be applied to obtain some new equilibrium theorems for abstract economies.

Keywords: Fixed point theorem, \(w\)-upper semicontinuous correspondences, correspondences with \(e\)-USS-property, abstract economy, equilibrium.

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1. Introduction

The pioneer work of Nash [10] first proved a theorem of equilibrium existence for games where the player’s payoffs are represented by continuous quasi-concave utilities. Arrow and Debreu used the work by Nash to prove the existence of equilibrium in a generalized N-person game or on abstract economy [7] which implies the Walrasian equilibrium existence. These ideas were extended by various authors in several ways. In [13], W. Shafer and H. Sonnenschein proved the existence of equilibrium of an economy with finite dimensional commodity space and irreflexive preferences represented as correspondences with open graph. Yannelis and Prabhakar [18] developed new techniques based on selection theorems and fixed-point theorems. Their main
result concerns the existence of equilibrium when the constraint and preference correspondences have open lower sections. They work within different frameworks (countable infinite number of agents, infinite dimensional strategy spaces). K. J. Arrow and G. Debreu proved the existence of Walrasian equilibrium in [2].

A. Borglin and H. Keiding [3] used new concepts of K.F.-correspondences and KF-majorized correspondences for their existence results. The concept of KF-majorized correspondences was extended by Yannelis and Prabhakar [18] to L-majorized correspondences. In [19], Yuan proposed a more general model of abstract economy than the one introduced by Borglin and Keing in [3], in the sense that the constraint mapping was split into two parts $A$ and $B$. This is due to the "small" constraint correspondence $A$ which could not have enough fixed points even though the "big" constraint correspondence $B$ could.

Most existence theorems of equilibrium deal with preference correspondences which have lower open sections or are majored by correspondences with lower open sections. In the last few years, some existence results were obtained for lower semicontinuous and upper semicontinuous correspondences. Some recent results concerning upper semicontinuous correspondences and fixed points can be found in [4], [14], [16], [19], [20].

In this paper, we define two types of correspondences: $w$-upper semicontinuous correspondences and correspondences that have e-USS-property. We prove a fixed point theorem for $w$-upper semicontinuous correspondences. This result is a Wu like result [17] and generalizes the Himmelberg's fixed point theorem in [9]. We use this theorem for proving our first theorem of equilibrium existence for abstract economies having $w$-upper semicontinuous constraint and preference correspondences. On the other hand, we use a technique of approximation to prove an equilibrium existence theorem for correspondences having e-USS-property.

The paper is organized in the following way: Section 2 contains preliminaries and notations. The fixed point theorem is presented in section 3 and the equilibrium theorems are stated in section 4.

2. Preliminaries and notation

Throughout this paper, we shall use the following notations and definitions:
Let $A$ be a subset of a topological space $X$. $\mathcal{F}(A)$ denotes the family of all nonempty finite subsets of $A$. $2^A$ denotes the family of all subsets of $A$. $\text{cl}A$ denotes the closure of $A$ in $X$. If $A$ is a subset of a vector space, $\text{co}A$ denotes the convex hull of $A$. If $F, G : X \to 2^Y$ are correspondences, then $\text{co}G, \text{cl}G, G \cap F : X \to 2^Y$ are correspondences defined by $(\text{co}G)(x) = \text{co}G(x), (\text{cl}G)(x) = \text{cl}G(x)$ and $(G \cap F)(x) = G(x) \cap F(x)$ for each $x \in X$, respectively. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$.

The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Gr}T\}$ (the set $\text{cl}_{X \times Y} \text{Gr}(T)$ is called the adherence of the graph of $T$). It is easy to see that $\text{cl}T(x) \subseteq \overline{T}(x)$ for each $x \in X$. Let $A$ be a subset of a topological space $X$.

Let $X, Y$ be topological spaces and $T : X \to 2^Y$ be a correspondence. $T$ is said to be upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subseteq V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(x) \subseteq V$ for each $y \in U$. $T$ is said to be almost upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subseteq V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(x) \subseteq \text{cl}V$ for each $y \in U$.

**Lemma 1.** (Lemma 3.2, pag. 94 in [21]) Let $X$ be a topological space, $Y$ be a topological linear space, and let $A : X \to 2^Y$ be an upper semicontinuous correspondence with compact values. Assume that the sets $C \subseteq Y$ and $K \subseteq Y$ are closed and respectively compact. Then $T : X \to 2^Y$ defined by $T(x) = (A(x) + C) \cap K$ for all $x \in X$ is upper semicontinuous.

Lemma 2 is a version of Lemma 1.1 in [20] (for $D = Y$, we obtain Lemma 1.1 in [20]). For the reader’s convenience, we include its proof below.

**Lemma 2.** Let $X$ be a topological space, $Y$ be a nonempty subset of a locally convex topological vector space $E$ and $T : X \to 2^Y$ be a correspondence. Let $\beta$ be a basis of neighbourhoods of 0 in $E$ consisting of open absolutely convex symmetric sets. Let $D$ be a compact subset of $Y$. If for each $V \in \beta$, the correspondence $T^V : X \to 2^Y$ is defined by $T^V(x) = (T(x) + V) \cap D$ for each $x \in X$, then $\cap_{V \in \beta} T^V(x) \subseteq T(x)$ for every $x \in X$.

**Proof.** Let be $x$ and $y$ be such that $y \in \cap_{V \in \beta} T^V(x)$ and suppose, by way of contradiction, that $y \notin \overline{T}(x)$. This means that $(x, y) \notin \text{cl} \text{Gr}T$, so that there exists an open neighborhood $U$ of $x$ and $V \in \beta$ such that:
\[(U \times (y + V)) \cap \text{Gr} T = \emptyset.\] 
Choose \(W \in \mathcal{B}\) such that \(W - W \subseteq V\) (e.g. \(W = \frac{1}{2} V\)). Since \(y \in T^W(x)\), then \((x, y) \in \text{clGr} T^W\), so that 
\[(U \times (y + W)) \cap \text{Gr} T^W \neq \emptyset.\]

There are some \(x' \in U\) and \(w' \in W\) such that \((x', y + w') \in \text{Gr} T^W\), i.e. \(y + w' \in T^W(x')\). Then, \(y + w' \in D\) and \(y + w' = y' + w''\) for some \(y' \in T(x')\) and \(w'' \in W\). Hence, \(y' = y + (w' - w'') \in y + (W - W) \subseteq y + V\), so that \(T(x') \cap (y + V) \neq \emptyset\). Since \(x' \in U\), this means that \((U \times (y + V)) \cap \text{Gr} T \neq \emptyset\), contradicting (1). \(\square\)

We introduce the following definitions.

**Definition 1.** The correspondence \(T : X \rightarrow 2^Y\) is said to be \(w\)-upper semicontinuous (weakly upper semicontinuous) with respect to the set \(D\) if there exists a basis \(\mathcal{B}\) of open symmetric neighborhoods of 0 in \(E\) such that, for each \(V \in \mathcal{B}\), the correspondence \(T^V\) is upper semicontinuous.

**Definition 2.** The correspondence \(T : X \rightarrow 2^Y\) is said to be almost \(w\)-upper semicontinuous (almost weakly upper semicontinuous) with respect to the set \(D\) if there exists a basis \(\mathcal{B}\) of open symmetric neighborhoods of 0 in \(E\) such that, for each \(V \in \mathcal{B}\), the correspondence \(\overline{T^V}\) is upper semicontinuous.

**Example 1.** Let \(T_1 : (0, 2) \rightarrow 2^{(0, 2)}\) be defined by \(T_1(x) = \begin{cases} (0, 1) \text{ if } x \in (0, 1]; \\ [1, 2) \text{ if } x \in (1, 2). \end{cases}\)

\(T_1\) and \(T_1 \cap \{1\} = \begin{cases} \emptyset \text{ if } x \in (0, 1]; \\ \{1\} \text{ if } x \in (1, 2) \end{cases}\) are not upper semicontinuous on \((0, 2)\).

Let \(D = \{1\}\) and let \(V = (-\varepsilon, \varepsilon), \varepsilon > 0\), be an open symmetric neighbourhood of 0 in \(\mathbb{R}\). Then, it results that 
for \(\varepsilon > 0\), 
\(T_1(x) + (-\varepsilon, \varepsilon) = \begin{cases} (-\varepsilon, 1 + \varepsilon) \text{ if } x \in (0, 1]; \\ (1 - \varepsilon, 2 + \varepsilon) \text{ if } x \in (1, 2); \end{cases}\)
\(T^V_1(x) = (T_1(x) + (-\varepsilon, \varepsilon)) \cap \{1\} = \{1\} \text{ for any } x \in (0, 2).\)
\(\overline{T^V_1}\) of \(x \in (0, 2)\).
For each \( V = (−\varepsilon, \varepsilon) \) with \( \varepsilon > 0 \), the correspondences \( T^V_1 \) and \( T^V_2 \) are upper semicontinuous and \( T^V_1 \) has nonempty values. We conclude that \( T_1 \) is \( w \)-upper semicontinuous with respect to \( D \) and it is also almost \( w \)-upper semicontinuous with respect to \( D \).

We also define the dual \( w \)-upper semicontinuity with respect to a compact set.

**Definition 3.** Let \( T_1, T_2 : X \to 2^Y \) be correspondences. The pair \( (T_1, T_2) \) is said to be dual almost \( w \)-upper semicontinuous (dual almost weakly upper semicontinuous) with respect to the set \( D \) if there exists a basis \( \beta \) of open symmetric neighborhoods of \( 0 \) in \( E \) such that, for each \( V \in \beta \), the correspondence \( T^V_{(1,2)} : X \to 2^D \) is lower semicontinuous, where \( T^V_{(1,2)} : X \to 2^D \) is defined by

\[
T^V_{(1,2)}(x) = (T_1(x) + V) \cap T_2(x) \cap D \quad \text{for each} \quad x \in X.
\]

**Example 2.** Let \( D = [1, 2], T_1 : (0, 2) \to 2^{[1,4]} \) be the correspondence defined by

\[
T_1(x) = \begin{cases} [2 - x, 2], & \text{if } x \in (0, 1); \\ \{4\}, & \text{if } x = 1; \\ [1, 2], & \text{if } x \in (1, 2). 
\end{cases}
\]

and \( T_2 : (0, 2) \to 2^{[2,3]} \) be the correspondence defined by

\[
T_2(x) = \begin{cases} [2, 3], & \text{if } x \in (0, 1); \\ \{2\}, & \text{if } x \in (1, 2). 
\end{cases}
\]

The correspondence \( T_1 \) is not upper semicontinuous on \( (0, 2) \).

For \( \varepsilon \in (0, 2] \), \( (T_1(x) + (−\varepsilon, \varepsilon)) \cap D \cap T_2(x) = \begin{cases} \{2\}, & \text{if } x \in (0, 1) \cup (1, 2); \\ \phi, & \text{if } x = 1. \end{cases} \)

For \( \varepsilon \in (2, \infty) \), \( (T_1(x) + (−\varepsilon, \varepsilon)) \cap D \cap T_2(x) = \{2\} \) for each \( x \in (0, 2) \).

Then, we have that for each \( \varepsilon > 0 \), \( T^V_{(1,2)}(x) = \{2\} \) for each \( x \in [0, 2] \) and the correspondence \( T^V_{(1,2)} \) is upper semicontinuous and has nonempty values.

We conclude that the pair \( (T_1, T_2) \) is dual almost \( w \)-upper semicontinuous with respect to \( D \).

### 3. A new fixed point theorem

We obtain the following fixed point theorem which generalizes Himmelberg’s fixed point theorem in [9]:
Theorem 3. Let $I$ be an index set. For each $i \in I$, let $X_i$ be a nonempty convex subset of a Hausdorff locally convex topological vector space $E_i$, $D_i$ be a nonempty compact convex subset of $X_i$ and $S_i, T_i : X := \prod_{i \in I} X_i \to 2^{X_i}$ be two correspondences with the following conditions:

1) for each $x \in X$, $S_i(x) \subset T_i(x)$.

2) $S_i$ is almost w-upper semicontinuous with respect to $D_i$ and $S_i^{\ast i}$ is convex nonempty valued for each absolutely convex symmetric neighborhood $V_i$ of $0$ in $E_i$.

Then there exists $x^* \in D := \prod_{i \in I} D_i$ such that $x^*_i \in T_i(x^*)$ for each $i \in I$.

Proof. Since $D_i$ is compact, $D := \prod_{i \in I} D_i$ is also compact in $X$. For each $i \in I$, let $\beta_i$ be a basis of open absolutely convex symmetric neighborhoods of zero in $E_i$ and let $\beta = \prod_{i \in I} \beta_i$. For each system of neighborhoods $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$, let’s define the correspondences $S_i^{V_i} : X \to 2^{D_i}$, by $S_i^{V_i}(x) = (S_i(x) + V_i) \cap D_i$, $x \in X$, $i \in I$. By assumption 2) each $S_i^{V_i}$ is u.s.c with nonempty closed convex values. Let’s define $S^V : X \to 2^D$ by $S^V(x) = \prod_{i \in I} S_i^{V_i}(x)$ for each $x \in D$.

The correspondence $S^V$ is upper semicontinuous with closed convex values. Therefore, according to Himmelberg’s fixed point theorem [9], there exists $x^*_i = \prod_{i \in I} x^*_i \in D$ such that $x^* \in S^V(x^*)$. It follows that $x^*_i \in S_i^{V_i}(x^*_i)$ for each $i \in I$.

For each $V = (V_i)_{i \in I} \in \beta$, let’s define $Q_V = \cap_{i \in I} \{x \in D : x_i \in S_i^{V_i}(x_V)\}$.

$Q_V$ is nonempty since $x^*_i \in Q_V$, then $Q_V$ is nonempty and closed.

We prove that the family $\{Q_V : V \in \beta\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of $\beta$ and let $V^{(k)} = \prod_{i \in I} V_i^{(k)}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \cap_{k=1}^n V_i^{(k)}$, then $V_i \in \beta_i$; thus $V = \prod_{i \in I} V_i \in \prod_{i \in I} \beta_i$.

Clearly $Q_V \subset \cap_{k=1}^n Q_{V^{(k)}}$ so that $\cap_{k=1}^n Q_{V^{(k)}} \neq \emptyset$.

Since $D$ is compact and the family $\{Q_V : V \in \beta\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in \beta\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in \beta\}$, then for each $V_i \in \beta_i$, $x^*_i \in S_i^{V_i}(x^*)$. According to Lemma 2, we have that $x^*_i \in S_i(x^*_i)$, for each $i \in I$, therefore $x^*_i \in T_i(x^*)$. □

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If $|I| = 1$ we get the result below.

**Corollary 4.** Let $X$ be a nonempty subset of a Hausdorff locally convex topological vector space $F$, $D$ be a nonempty compact convex subset of $X$ and $S, T : X \to 2^X$ be two correspondences with the following conditions:

1) for each $x \in X$, $\overline{S(x)} \subset T(x)$ and $S(x) \neq \emptyset$,
2) $S$ is almost $w$-upper semicontinuous with respect to $D$ and $S^V$ is convex valued for each open absolutely convex symmetric neighborhood $V$ of $0$ in $E$.

Then, there exists a point $x^* \in D$ such that $x^* \in T(x^*)$.

In the particular case that the correspondence $S = T$ the following result stands.

**Corollary 5.** Let $X$ be a nonempty subset of a Hausdorff locally convex topological vector space $F$, $D$ be a nonempty compact convex subset of $X$ and $T : X \to 2^X$ be an almost $w$- upper semicontinuous correspondence with respect to $D$ and $T^V$ is convex valued for each open absolutely convex symmetric neighborhood $V$ of $0$ in $E$. Then, there exists a point $x^* \in D$ such that $x^* \in T(x^*)$.

### 4. Applications in equilibrium theory

#### 4.1. The model of an abstract economy

Let $I$ be a nonempty set (the set of agents). For each $i \in I$, let $X_i$ be a non-empty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let $A_i, B_i : X \to 2^{X_i}$ be the constraint correspondences and $P_i$ the preference correspondence.

**Definition 4.** An abstract economy $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ is defined as a family of ordered quadruples $(X_i, A_i, P_i, B_i)$.

**Definition 5 (19).** An equilibrium for $\Gamma$ is defined as a point $x^* \in X$ such that for each $i \in I$, $x_i^* \in \overline{B_i(x^*)}$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. 

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Remark 1. When, foreach $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$, the abstract economy model coincides with the classical one introduced by Borglin and Keiding in [3]. If in addition, $\overline{B_i(x^*)} = \text{cl}_{X_i} B_i(x^*)$ for each $x \in X$, which is the case where $B_i$ has a closed graph in $X \times X_i$, the definition of equilibrium coincides with the one used by Yannelis and Prabhakar in [18].

Remark 2. If the preference correspondence $P_i$ is defined by using a utility function $u_i$, that is $P_i(x) = \{y \in X_i : u_i(y) > u_i(x_i)\}$, the irreflexibility condition $x_i \notin \overline{P_i}(x)$ which appears among the hypothesis of the existence equilibrium theorems may fail. A case in which this condition is verified, is when $P_i$ is an order interval preference. Order interval preferences are studied in Chateauneuf [5]. These preference relations $\prec$ (on $X$) are representable, if two real valued functions $u$ and $v$ on $X$ exist and are such that: $x \prec y \iff u(x) < v(y)$. If a representation of the preference relation $\prec_i$ exist, we can define the preference correspondence $P_i$ by $P_i(x) = \{y \in X_i : v_i(y) > u_i(x_i)\}$ and the condition $x_i \notin \overline{P_i}(x)$ can be fulfilled.

4.2. Examples of abstract economies with two constraint correspondences

A first example of an abstract economy with two constraint correspondences is the one associated to the model proposed by Radner in [11] and his followers. This is the model of a pure exchange economy with asymmetric information. In Radner model there is a finite number of possible future states of nature. Each agent has his own private, and typically incomplete, information about the future state of nature. Let $I = \{1, 2, ..., n\}$ be the set of agents and $l$ the number of commodities on the market. Uncertainty is described by $\Omega = \{s_1, s_2, ..., s_m\}$, the set of the future states of the nature. For each agent $i \in I$, the initial private information is a partition on $\Omega$, induced by a signal $\pi_i : \Omega \rightarrow Y_i$. Radner assumes two periods of time for the economy, present and future. In Radner ([11]) the initial information is kept fixed. In Radner ([12]) agents forecast the future and at the end of the first period, the initial information of each agent is updated with a signal given by prices $p$. Finally, a more refined partition of $\Omega$ is obtained as the joint of the initial information and the information generated by $\hat{\pi}_i(p) : \Omega \rightarrow Y_i$.

For agent $i \in I$, the consumption plan in the first period will be denoted by $x^{i}_0 \in \mathbb{R}_{+}^l$ and in the second period, for each state $s_j$, $j = 1, 2, ..., m$, it will be denoted by $x^{i}_j \in \mathbb{R}_{+}^l$. A bundle for agent $i$ is $x^i = (x^i_0, x^i_1, x^i_2, ..., x^i_m)$. Let $X_i = \mathbb{R}_{+}^{nl+1}$. Each agent has a preference correspondence $Q_i : \prod_{i \in I} X_i \rightarrow 2^{X_i}$ and an initial endowment $e^i = (e^i_0, e^i_1, e^i_2, ..., e^i_m) \in X_i$. 


Definition 6. A pure exchange economy with asymmetric information is the family $\mathcal{E} = (I, \Omega, \pi_i, Q'_i, e^i)_{i \in I}$.

Definition 7. An allocation for the economy is $x = (x^i)_{i \in I}$. The allocation is called physically feasible if $\sum_{i \in I} x^i \leq \sum_{i \in I} e^i$ and informationally feasible for each agent $i$ if $\pi_i(p)(s) = \pi_i(p)(s')$ implies $x^i_s = x^i_{s'}$.

Let $p_0$ be the price in the first period, for the second period let $p_j$ be the price in the state $j$, $j = 1, 2, ..., m$ and let $p = (p_0, p_1, ..., p_m)$. Let $\Delta$ be the normalized set of prices.

The budget correspondence of agent $i$ is $B_i : \Delta \to \mathbb{R}_{+}^{lm+1}$, defined by

$$B_i(p) = \{ x^i \in \mathbb{R}_{+}^{lm+1} : px^i < pe^i \}.$$  

The information correspondence of agent $i$ is $I_i : \Delta \to \mathbb{R}_{+}^{lm+1}$, defined by

$$I_i(p) = \{ x^i \in \mathbb{R}_{+}^{lm+1} : x^i_s = x^i_{s'} \text{ if } \pi_i(p)(s) = \pi_i(p)(s') \}.$$  

Definition 8. The pair $(x^*, p^*) \in \mathbb{R}_{+}^{n(lm+1)} \times \Delta$ is a Radner equilibrium for the economy $\mathcal{E}$ if

1) $\sum_{i \in I} (x^*)^i \leq \sum_{i \in I} e^i$

and for each $i \in I$,

2) $(x^*)^i \in I_i(p^*) \cap B_i(p^*)$;

3) $y^i \in Q'_i(x^*) \cap I_i(p^*)$ implies that $y^i \not\in B_i(p^*)$.

Let $X = \prod_{i \in I} X_i \times \Delta$, where for $i \in I$, $X_i = \mathbb{R}_{+}^{lm+1}$ is the consumption set of agent $i$. Let’s define the following correspondences:

- for each $i \in I$, $Q_i : X \to 2^{X_i}$ is the preference correspondence defined by $Q_i(x, p) = Q'_i(x)$ for each $(x, p) \in X$;

- $Q_{n+1} : X \to 2^\Delta$ is the preference correspondence defined by $Q_{n+1}(x, p) = \{ q \in \Delta : q(\sum_{i \in I} (x^i - e^i)) > p(\sum_{i \in I} (x^i - e^i)) \}$ for each $(x, p) \in X$;

- for $i \in I$, $A_i : X \to 2^{X_i}$ is defined by $A_i(x, p) = \{ y^i \in \mathbb{R}_{+}^{lm+1} : py^i < pe^i \}$ for each $(x, p) \in X$;

- $A_{n+1} : X \to 2^\Delta$ is defined by $A_{n+1}(x, p) = \Delta$ for each $(x, p) \in X$;

- for $i \in I$, $I_i : X \to 2^X_i$ is defined by $I_i(x, p) = \{ y^i \in \mathbb{R}_{+}^{lm+1} : y^i_s = y^i_{s'} \text{ if } \pi_i(p)(s) = \pi_i(p)(s') \}$ for each $(x, p) \in X$;

- $I_{n+1} : X \to 2^\Delta$ is defined by $I_{n+1}(x, p) = \Delta$ for each $(x, p) \in X$.
Definition 9. The abstract economy associated to the model of the pure exchange economy with asymmetric information is $\Gamma = (X_i, A_i, P_i, B_i)_{i \in \{1, \ldots, n+1\}}$, where:

- for $i \in I$, $X_i = \mathbb{R}_{i}^{m+1}$ is the consumption set of agent $i$ and let $X = \prod_{i \in I} X_i \times \Delta$;
- $P_i : X \to 2^{X_i} (i \in I)$ and $P_{n+1} : X \to 2^\Delta$ are the preference correspondences defined by $P_i(x, p) = Q_i(x, p) \cap I_i(x, p)$ for each $(x, p) \in X$ and $i \in \{1, 2, \ldots, n+1\}$;
- $A_i : X \to 2^{X_i} (i \in I)$ and $A_{n+1} : X \to 2^\Delta$ are the constraint correspondences defined above;
- $B_i : X \to 2^{X_i} (i \in I)$ and $B_{n+1} : X \to 2^\Delta$ are the constraint correspondences defined by $B_i(x, p) = A_i(x, p) \cap I_i(x, p)$ for each $(x, p) \in X$ and $i \in \{1, 2, \ldots, n+1\}$.

Remark 3. We note that $A_i(x, p) \cap P_i(x, p) \subseteq B_i(x, p)$ for each $(x, p) \in X$ and for each $i \in \{1, 2, \ldots, n+1\}$.

Proposition 6. An equilibrium for the associated abstract economy $\Gamma$ is a Radner equilibrium of the economy with asymmetric information $\mathcal{E}$.

Proof. Let $(x^*, p^*)$ be an equilibrium for $\Gamma$.

1) For each $i \in \{1, 2, \ldots, n\}$, we have that $(x^*)_i \in B_i(x^*, p^*) = A_i(x^*, p^*) \cap I_i(x^*, p^*)$ and thus, by definition of $A_i$ and $I_i$, $(x^*)_i \in I_i(p^*) \cap B_i(p^*)$;

2) $p^* \in B_{n+1}(x^*, p^*) = \Delta$;

3) for each $i \in \{1, 2, \ldots, n\}$, we have that $A_i(x^*, p^*) \cap P_i(x^*, p^*) = \phi$, which implies that if $y_i^* \in P_i(x^*, p^*) = Q_i(x^*, p^*) \cap I_i(x^*, p^*)$, then $y_i^* \notin A_i(x^*, p^*)$.

This means that $y_i^* \in Q_i'(x^*) \cap I_i(p^*)$ implies that $y_i^* \notin B_i(p^*)$;

4) we have that $A_{n+1}(x^*, p^*) \cap P_{n+1}(x^*, p^*) = \phi$, which is equivalent with

$$\{q \in \Delta : q(\sum_{i \in I} ((x^*)_i - e^i)) > p^*(\sum_{i \in I} ((x^*)_i - e^i))\} \cap \Delta = \phi.$$  
This fact implies that $q(\sum_{i \in I} ((x^*)_i - e^i)) \leq p^*(\sum_{i \in I} ((x^*)_i - e^i))$ for all $q \in \Delta$. If we choose $q$ as a vector of the canonical basis of $\mathbb{R}^{ml+1}$, that is $q_j = 1$ and $q_i = 0$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, ml+1\}$, we obtain that $\sum_{i \in I} (x^*)_i \leq \sum_{i \in I} e^i$. $\Box$

The second example is the abstract economy associated to an exchange economy with two constraint correspondences, the first one being the budget
correspondence and the second one being the consumption set that depends on prices.

The third example follows the idea of an exchange economy which has, beyond the budget correspondence, a second constraint correspondence $D_i$, defined by the delivery conditions as stated in the paper by Correia-da-Silva and Herves-Beloso [6].

Let’s assume that the set of the states of nature is $\Omega = \{1, 2, \ldots, m\}$, the future prices are $p_1, p_2, \ldots, p_m \in \mathbb{R}^+_l$, and that each agent $i$ has a signal $f_i : \Omega \to Y_i$ such that $f_i(s) = f_i(s')$ if $s$ and $s'$ are states that cannot be distinguished. The agent $i$ chooses a portfolio $y(s)$ in the following way:

\[ p_s y(s) \leq p_s y(s') \text{ for all } s' \text{ such that } f_i(s') = f_i(s). \]

The correspondence $D_i : X \times \Delta \to 2^{\mathbb{R}^m}$ is defined by $D_i(x, p) = \{ y \in \mathbb{R}^m : p_s y(s) \leq p_s y(s') \text{ for all } s' \text{ such that } f_i(s') = f_i(s) \}$. 

4.3. The existence of equilibria in locally convex spaces

As an application of the fixed point Theorem 3, we have the following result.

**Theorem 7.** Let $\Gamma = \{X_i, A_i, B_i, P_i\}_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled:

1) $X_i$ is a nonempty convex subset of a Hausdorff locally convex topological vector space $E_i$ and $D_i$ is a nonempty compact convex subset of $X_i$;

2) for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ and $P_i(x)$ are convex, $B_i(x)$ is nonempty, convex and $A_i(x) \cap P_i(x) \subset B_i(x)$;

3) $W_i = \{ x \in X : A_i(x) \cap P_i(x) \neq \emptyset \}$ is open in $X$.

4) $H_i : X \to 2^{X_i}$ defined by $H_i(x) = A_i(x) \cap P_i(x)$ for each $x \in X$ is almost w-upper semicontinuous with respect to $D_i$ on $W_i$ and $\overline{H_i V_i}$ is convex nonempty valued for each open absolutely convex symmetric neighborhood $V_i$ of 0 in $E_i$;

5) $B_i : X \to 2^{X_i}$ is almost w-upper semicontinuous with respect to $D_i$ and $\overline{B_i V_i}$ is convex nonempty valued for each open absolutely convex symmetric neighborhood $V_i$ of 0 in $E_i$;

6) for each $x \in X$, $x_i \not\in (A_i \cap P_i)(x)$;

Then there exists $x^* \in D = \prod_{i \in I} D_i$ such that $x_i^* \in B_i(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$ for each $i \in I$. 

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Proof. Let \( i \in I \). By condition (3) we know that \( W_i \) is open in \( X \).

Let’s define \( T_i : X \to 2^{X_i} \) by \( T_i(x) = \begin{cases} \ A_i(x) \cap P_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \notin W_i \end{cases} \) for each \( x \in X \).

Then \( T_i : X \to 2^{X_i} \) is a correspondence with nonempty convex values. We shall prove that \( T_i : X \to 2^{D_i} \) is almost w-upper semicontinuous with respect to \( D_i \). Let \( \beta_i \) be a basis of open absolutely convex symmetric neighborhoods of 0 in \( E_i \) and let \( \beta = \prod \beta_i \).

For each \( V = (V_i)_{i \in I} \in \prod \beta_i \), for each \( x \in X \), let for each \( i \in I \)

\[
B^{V_i}(x) = (B_i(x) + V_i) \cap D_i, \\
F^{V_i}(x) = ((A_i(x) + P_i(x)) + V_i) \cap D_i \text{ and} \\
T_i^{V_i}(x) = \begin{cases} F^{V_i}(x), & \text{if } x \in W_i, \\ B^{V_i}(x), & \text{if } x \notin W_i. \end{cases}
\]

For each open set \( V'_i \) in \( D_i \), the set

\[
\left\{ x \in X : T_i^{V_i}(x) \subset V'_i \right\} = \\
\left\{ x \in W_i : F^{V_i}(x) \subset V'_i \right\} \cup \left\{ x \in X \setminus W_i : B^{V_i}(x) \subset V'_i \right\}
\]

According to condition (4), the set \( \left\{ x \in W_i : F^{V_i}(x) \subset V'_i \right\} \) is open in \( X \).

The set \( \left\{ x \in X : B^{V_i}(x) \subset V'_i \right\} \) is open in \( X \) because \( B^{V_i} \) is upper semicontinuous.

Therefore, the set \( \left\{ x \in X : T_i^{V_i}(x) \subset V'_i \right\} \) is open in \( X \). It shows that \( T_i^{V_i} : X \to 2^{D_i} \) is upper semicontinuous. According to Theorem 3, there exists \( x^* \in D = \prod_{i \in I} D_i \) such that \( x^* \in T_i(x^*) \), for each \( i \in I \). By condition (5) we have that \( x_i^* \in D_i(x^*) \) and \( (A_i \cap P_i)(x^*) = \emptyset \) for each \( i \in I \).

Example 3. Let \( \Gamma = \{X_i, A_i, B_i, P_i\}_{i \in I} \) be an abstract economy, where \( I = \{1, 2, ..., n\} \), \( X_i = [0;4] \) be a compact convex choice set, \( D_i = [0,2] \) for each \( i \in I \) and \( X = \prod_{i \in I} X_i \).

Let the correspondences \( A_i, B_i, P_i : X \to 2^{X_i} \) be defined as follows:

for each \((x_1, x_2, ..., x_n) \in X\),
\[ A_i(x) = \begin{cases} 
[1 - x_i, 2] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in (0, \frac{1}{2}); \\
[1 - x_i, 2) & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [\frac{1}{2}, 1); \\
[3, 4] & \text{if } x = 0; \\
[0, \frac{1}{2}], & \text{otherwise;}
\end{cases} \]

\[ P_i(x) = \begin{cases} 
[\frac{3}{2}, 2 + x_i] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [0, 1); \\
(1, 2], & \text{otherwise;}
\end{cases} \]

\[ B_i(x) = \begin{cases} 
[0, 2] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [0, 1); \\
[3, 4] & \text{if } x = 0; \\
[0, 2], & \text{otherwise.}
\end{cases} \]

The correspondences \( A_i, B_i, P_i \) are not upper semicontinuous on \( X \).

\[ A_i(x) \cap P_i(x) = \begin{cases} 
[\frac{3}{2}, 2] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in (0, \frac{1}{2}); \\
[\frac{3}{2}, 2) & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [\frac{1}{2}, 1); \\
\phi, & \text{otherwise.}
\end{cases} \]

\[ W_i = \{ x \in X : A_i(x) \cap P_i(x) \neq \emptyset \} = (0, 1)^n \text{ is open in } X. \]

\[ (A_i \cap P_i)(x) = \begin{cases} 
[\frac{3}{2}, 2] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [0, 1]; \\
\phi, & \text{otherwise.}
\end{cases} \]

We notice that for each \( x \in X, x_i \notin (A_i \cap P_i)(x) \).

We shall prove that \( B_i \) and \( (A_i \cap P_i)_W \) are almost \( w \)-upper semicontinuous with respect to \( D_i = [0, 2] \).

On \( W_i \),

\[ (A_i \cap P_i)(x) = \begin{cases} 
[\frac{3}{2}, 2] & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in (0, \frac{1}{2}); \\
[\frac{3}{2}, 2) & \text{if } x \in X \text{ and } \forall j \in \{1, 2, \ldots, n\}, \ x_j \in [\frac{1}{2}, 1); \\
\phi, & \text{otherwise;}
\end{cases} \]

\[ (A_i \cap P_i)(x) + (-\varepsilon, \varepsilon) = (\frac{3}{2} - \varepsilon, 2 + \varepsilon) \text{ if } x \in X, \forall j \in \{1, 2, \ldots, n\}, \ x_j \in (0, 1); \]

Let \( (A_i \cap P_i)^V(x) = ((A_i \cap P_i)(x) + (-\varepsilon, \varepsilon)) \cap [0, 2] \), where \( V = (-\varepsilon, \varepsilon) \).

Then,

if \( \varepsilon \in (0, \frac{3}{2}] \),

\[ (A_i \cap P_i)^V(x) = (\frac{3}{2} - \varepsilon, 2] \text{ if } x \in (0, 1)^n; \]

if \( \varepsilon > \frac{3}{2} \),

\[ (A_i \cap P_i)^V(x) = [0, 2] \text{ if } x \in (0, 1)^n; \]

Hence, for each \( V = (-\varepsilon, \varepsilon), (A_i \cap P_i)^V_{W_i} \) is upper semicontinuous and has nonempty values.

\[ B_i(x) + (-\varepsilon, \varepsilon) = \begin{cases} 
(-\varepsilon, 2 + \varepsilon) & \text{if } x \in X, \forall j \in \{1, 2, \ldots, n\}, \ x_j \in (0, 1); \\
(3 - \varepsilon, 4 + \varepsilon) & \text{if } x = 0; \\
(-\varepsilon, 2 + \varepsilon) & \text{otherwise.}
\end{cases} \]

Let \( B_i^V(x) = (B_i(x) + (-\varepsilon, \varepsilon)) \cap [0, 2], \) where \( V = (-\varepsilon, \varepsilon) \).

Then,

if \( \varepsilon \in (0, 1), \)
$B_i^V(x) = \begin{cases} \phi & \text{if } x = 0; \\
[0,2] & \text{otherwise;} \\
[0,2] & \text{if } x \in X \text{ and } \forall j \in \{1,2,...,n\}, x_j \in [0,1); \\
(3-\varepsilon,2] & \text{if } x = 0; \\
[0,2] & \text{otherwise.} \\
\end{cases}$

and if $\varepsilon > 3$, $B_i^V(x) = [0,2]$ if $x \in X$.

Then, for each $V = (-\varepsilon,\varepsilon)$, $B_i^V$ is upper semicontinuous and has non-empty values.

Therefore, all hypotheses of Theorem 2 are satisfied, so that there exist equilibrium points. For example, $x^* = \{\frac{3}{2},\frac{3}{2},...\} \in X$ verifies $x^*_i \in B_i(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$. □

Theorem 8 deals with abstract economies which have dual w-upper semicontinuous pairs of correspondences.

**Theorem 8.** Let $\Gamma = \{X_i,A_i,B_i,P_i\}_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled:

1) $X_i$ is a nonempty convex subset of a Hausdorff locally convex topological vector space $E_i$ and $D_i$ is a nonempty compact convex subset of $X_i$;
2) for each $x \in X = \prod_{i \in I} X_i$, $P_i(x) \subset D_i$, $A_i(x) \cap P_i(x) \subset B_i(x)$ and $B_i(x)$ is nonempty;
3) the set $W_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in $X$;
4) the pair $(A_i|_{W_i},P_i|_{W_i})$ is dual almost w-upper semicontinuous with respect to $D_i$, $B_i : X \to 2^{X_i}$ is almost w-upper semicontinuous with respect to $D_i$;
5) if $T_{i,V_i} : X \to 2^{X_i}$ is defined by $T_{i,V_i}(x) = (A_i(x) + V_i) \cap D_i \cap P_i(x)$ for each $x \in X$, then the correspondences $B_i^{V_i}$ and $T_{i,V_i}$ are nonempty convex valued for each open absolutely convex symmetric neighborhood $V_i$ of 0 in $E_i$;
6) for each $x \in X$, $x_i \notin P_i(x)$;

Then, there exists $x^* \in D = \prod_{i \in I} D_i$ such that $x^*_i \in B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I$.

**Proof.** For each $i \in I$, let $\beta_i$ denote the family of all open absolutely convex symmetric neighborhoods of zero in $E_i$ and let $\beta = \prod_{i \in I} \beta_i$. For each $V = \prod_{i \in I} V_i \in \prod_{i \in I} \beta_i$, for each $i \in I$, let
\[ B_{iV}(x) = (B_i(x) + V_i) \cap D_i \] for each \( x \in X \) and
\[ S_{iV}(x) = \begin{cases} T_{iV}(x), & \text{if } x \in W_i, \\ B_{iV}(x), & \text{if } x \notin W_i, \end{cases} \]

\( S_{iV} \) has closed values. Next, we shall prove that \( S_{iV}: X \rightarrow 2^{D_i} \) is upper semicontinuous.

For each open set \( V' \) in \( D_i \), the set
\[ \{ x \in X : S_{iV}(x) \subset V' \} = \]
\[ = \{ x \in W_i : T_{iV}(x) \subset V' \} \cup \{ x \in X \setminus W_i : B_{iV}(x) \subset V' \} \]
\[ = \{ x \in W_i : T_{iV}(x) \subset V' \} \cup \{ x \in X : B_{iV}(x) \subset V' \}. \]

We know that the correspondence \( T_{iV}(x)|_{W_i} : W_i \rightarrow 2^{D_i} \) is upper semicontinuous. The set \( \{ x \in W_i : T_{iV}(x) \subset V' \} \) is open in \( X \). Since \( B_{iV}(x) : X \rightarrow 2^{D_i} \) is upper semicontinuous, the set \( \{ x \in X : B_{iV}(x) \} \subset V' \) is open in \( X \) and therefore, the set \( \{ x \in X : S_{iV}(x) \subset V' \} \) is open in \( X \). It proves that \( S_{iV}: X \rightarrow 2^{D_i} \) is upper semicontinuous. According to Himmelberg’s Theorem, applied for the correspondences \( S_{iV} \), there exists a point \( x^*_V \in D = \prod_{i \in I} D_i \) such that \( x^*_V \in S_{iV}(x^*_V) \) for each \( i \in I \). By condition (5), we have that
\[ x^*_V \notin \overline{P_i(x^*_V)}, \text{ hence, } x^*_V \notin A^*_V(x^*_V) \cap \overline{P_i(x^*_V)}. \]

We also have that \( \text{clGr}(T_{iV}) \subseteq \text{clGr}(A^*_V) \cap \text{clGr}P_i \). Then \( T_{iV}(x) \subseteq A^*_V(x) \cap \overline{P_i(x)} \) for each \( x \in X \). It follows that \( x^*_V \notin \overline{T_{iV}(x^*_V)} \). Therefore, \( x^*_V \in B_{iV}(x^*_V) \).

For each \( V = (V_i)_{i \in I} \subseteq \prod_{i \in I} \beta_i \), let’s define \( Q_V = \cap_{i \in I} \{ x \in D : x \in B_{iV}(x) \} \) and \( A_i(x) \cap P_i(x) = \emptyset \).

\( Q_V \) is nonempty since \( x^*_V \in Q_V \), and it is a closed subset of \( D \) according to (3). Then, \( Q_V \) is nonempty and compact.

Let \( \beta = \prod_{i \in I} \beta_i \). We prove that the family \( \{ Q_V : V \in \beta \} \) has the finite intersection property.

Let \( \{ V^{(1)}, V^{(2)}, ..., V^{(n)} \} \) be any finite set of \( \beta \) and let \( V^{(k)} = \prod_{i \in I} V_i^{(k)} \), \( k = 1, ..., n \). For each \( i \in I \), let \( V_i = \bigcap_{k=1}^{n} V_i^{(k)} \), then \( V_i \in \beta_i \); thus \( V \in \prod_{i \in I} \beta_i \).

Clearly \( Q_V \subseteq \bigcap_{k=1}^{n} Q_{V^{(k)}} \) so that \( \bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset \).
Since $D$ is compact and the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in \mathcal{B}\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in \mathcal{B}\}$, then for each $V \in \mathcal{B},$

$$x^* \in \bigcap_{i \in I} \left\{ x^* \in D : x_i^* \in \overline{B^V}(x) \text{ and } A_i(x) \cap P_i(x) = \emptyset \right\}.$$ 

Hence, $x_i^* \in \overline{B^V}(x^*)$ for each $V \in \mathcal{B}$ and for each $i \in I$. According to Lemma 2, we have that $x_i^* \in \overline{B_i}(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$ for each $i \in I$. \qed

We now introduce the following concept, which also generalizes the concept of lower semicontinuous correspondences.

**Definition 10.** Let $X$ be a non-empty convex subset of a topological linear space $E$, $Y$ be a non-empty set in a topological space and $K \subseteq X \times Y$.

The correspondence $T : X \times Y \to 2^X$ has the e-USCS-property (e-upper semicontinuous selection property) on $K$, if for each absolutely convex neighborhood $V$ of zero in $E$, there exists an upper semicontinuous correspondence with convex values $S^V : X \times Y \to 2^X$ such that $S^V(x, y) \subset T(x, y) + V$ and $x \notin \text{cl}S^V(x, y)$ for every $(x, y) \in K$.

The following theorem is an equilibrium existence result for economies with constraint correspondences having e-USCS-property.

**Theorem 9.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty compact convex set in a locally convex space $E_i$;
2. $\text{cl}B_i$ is upper semicontinuous with non-empty convex values;
3. the set $W_i := \{ x \in X / (A_i \cap P_i)(x) \neq \emptyset \}$ is open;
4. $\text{cl}(A_i \cap P_i)$ has the e-USCS-property on $W_i$.

Then there exists an equilibrium point $x^* \in X$ for $\Gamma$, i.e., for each $i \in I,$ $x_i^* \in \overline{B_i}(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

**Proof.** For each $i \in I$, let $\mathcal{B}_i$ denote the family of all open convex neighborhoods of zero in $E_i$. Let $V = (V_i)_{i \in I} \in \prod_{i \in I} \mathcal{B}_i$. Since $\text{cl}(A_i \cap P_i)$ has the e-USCS-property on $W_i$, it follows that there exists an upper semicontinuous correspondence $F_i^{V_i} : X \to 2^{X_i}$ such that $F_i^{V_i}(x) \subset \text{cl}(A_i \cap P_i)(x) + V_i$ and $x_i \notin \text{cl}F_i^{V_i}(x)$ for each $x \in W_i$.

Define the correspondence $T_i^{V_i} : X \to 2^{X_i}$, by

$$T_i^{V_i}(x) := \begin{cases} \text{cl}\{F_i^{V_i}(x)\}, & \text{if } x \in W_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i, & \text{if } x \notin W_i. \end{cases}$$
Let $U$ be an open subset of $X_i$, then
\[
U' := \{ x \in X \mid T_i^V(x) \subset U \}
\]
\[
= \{ x \in W_i \mid T_i^V(x) \subset U \} \cup \{ x \in X \setminus W_i \mid T_i^V(x) \subset U \}
\]
\[
= \{ x \in W_i \mid cl F_i^V(x) \subset U \} \cup \{ x \in X \setminus cl B_i(x) + cl V_i \subset U \}
\]

$U'$ is an open set, because $W_i$ is open, \( \{ x \in W_i \mid cl F_i^V(x) \subset U \} \) open since \( cl F_i^V(x) \) is an upper semicontinuous map on $W_i$ and the set \( \{ x \in X \mid (cl B_i(x) + cl V_i) \cap X_i \subset U \} \) is open since \( (cl B_i + cl V_i) \cap X_i \) is u.s.c. Then \( T_i^V \) is upper semicontinuous on $X$ and has closed convex values.

Define \( T^V : X \to 2^X \) by \( T^V(x) := \prod_{i \in I} T_i^V(x) \) for each \( x \in X \).

\( T^V \) is an upper semicontinuous correspondence and has also non-empty convex closed values.

Since $X$ is a compact convex set, by Fan’s fixed-point theorem [8], there exists \( x^*_V \in X \) such that \( x^*_V \in T^V(x^*_V) \), i.e., for each \( i \in I \), \( x^*_V \in T_i^V(x^*_V) \). If \( x^*_V \in W_i \), \( x^*_V \in cl F_i^V(x^*_V) \), which is a contradiction.

Hence, \( x^*_V \in cl B_i(x^*_V) + V_i \cap X_i \) and \( (A_i \cap P_i)(x^*_V) = \emptyset \), i.e. \( x^*_V \in Q_V \) where
\[
Q_V := \cap_{i \in I} \{ x \in X : x \in cl B_i(x) + V_i \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset \}.
\]

Since $W_i$ is open, $Q_V$ is the intersection of non-empty closed sets, therefore it is non-empty, closed in $X$.

We prove that the family \( \{ Q_V : V \in \prod_{i \in I} \beta_i \} \) has the finite intersection property.

Let \( \{ V^{(1)}, V^{(2)}, ..., V^{(n)} \} \) be any finite set of \( \prod_{i \in I} \beta_i \) and let \( V^{(k)} = (V_i^{(k)})_{i \in I}, k = 1, ..., n \). For each \( i \in I \), let \( V_i = \bigcap_{k=1}^n V_i^{(k)} \), then \( V_i \in \beta_i \); thus \( V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i \). Clearly \( Q_V \subset \bigcap_{k=1}^n Q_{V^{(k)}} \) so that \( \bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset \).

Since $X$ is compact and the family \( \{ Q_V : V \in \prod_{i \in I} \beta_i \} \) has the finite intersection property, we have that \( \cap \{ Q_V : V \in \prod_{i \in I} \beta_i \} \neq \emptyset \). Take any \( x^* \in \cap \{ Q_V : V \in \prod_{i \in I} \beta_i \} \neq \emptyset \). For each \( i \in I \) and each \( V_i \in \beta_i \), \( x^*_i \in cl B_i(x^*) + V_i \cap X_i \) and \( (A_i \cap P_i)(x^*) = \emptyset \); but then \( x^*_i \in cl B_i(x^*) \) from Lemma 2 and \( (A_i \cap P_i)(x^*) = \emptyset \) for each \( i \in I \) so that \( x^* \) is an equilibrium point of $\Gamma$ in $X$. \( \square \)
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