An undominated Nash equilibrium for voting by committees with exit

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Abstract: We consider the problem of a society whose members must choose, with a voting by committees, a subset of new members from a given set of candidates. After knowing the elected candidates, former members may reconsider their membership in the society by either staying or exiting. We analyze the voting behavior of members who take into account the effect of their votes not only on the elected candidates, but also on the final composition of the society. For a restricted domain (additive and monotonic preference profiles with dichotomous bads) we construct a (voting) strategy profile that is an undominated Nash equilibrium of the induced voting game.

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Keywords: Voting, Committees, Undominated Nash Equilibrium.

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1 Introduction

Societies use voting rules to make decisions. The elections of representatives in democratic societies, the public positions taken up by political parties on different issues, or the admission of new members in a society are some examples of this. For this last example, Barberà, Sonnenschein, and Zhou (1991) consider the problem where a finite set of members who originally make up a society has to decide which candidates, to be chosen from a given set, will be elected to become new members of the society. They assume that former members of the society cannot leave it as a result of its change in composition. But often, the entrance of new members trigger the exit of former ones. For the static setting where members cannot leave the society, they characterize voting by committees as the class of strategy-proof and onto social choice functions whenever members’ preferences over subsets of candidates are either separable or additively representable. However, strategy-proofness becomes too strong whenever the mentioned evolution of the society is explicitly considered. Then the study of strategic behavior requires the analysis of equilibria. This is the aim of this paper.

Three lines of research have already focused on this idea. A first one considers a society that, during a fixed and commonly known number of periods, may admit in each period a subset of new members. Within this dynamic setup, an interesting issue arises: voters, at earlier stages, vote not only according to whether or not they like a candidate but also according to their tastes concerning future candidates. Barberà, Maschler, and Shalev (2001) study the particular case where members have dichotomous preferences (candidates are either friends or enemies) and the voting rule used by the society is quota one (it is sufficient to receive one vote to be elected). They identify and study (subgame perfect and trembling-hand perfect) equilibria where members exhibit, due to the dynamics of the game, complex strategic voting behavior. Granot, Maschler, and Shalev (2002) study a similar model with expulsion; current members of the society have to decide each period whether to admit by unanimity new members into the society and whether to expel current members by others’ unanimity. They study equilibria for different protocols which depend on whether the expulsion decision has to be taken each period either simultaneously with, before, or after the admission decision.

In a second line of research, a set of voters and a set of candidates (which may overlap) must select a representative candidate (or a subset of them). The key issue this literature addresses is the incentives of candidates, given a particular voting
rule (how voters choose a candidate or a subset of candidates), to enter or exit the election in order to strategically affect the outcome of the rule. By imposing some independence conditions and an “internal stability” condition (the losing candidates must not have an incentive to drop out of the election) they prove that the class of voting rules immune to this strategic manipulation is only composed of dictatorial rules.\footnote{See Dutta, Jackson, and Le Breton (2001) for single-valued voting rules, and Ehlers and Weymark (2003), Eraslan and McLennan (2002), and Rodríguez-Álvarez (2003) for multi-valued voting rules.}

In this paper we contribute on a third line of research by considering explicitly the possibility that, in the Barberà, Sonnenschein, and Zhou’s (1991) setup members who originally conform a society have the option to leave it voluntarily. In Berga, Bergantiños, Massó, and Neme (2003a) we showed that the unique social choice function that is still strategy-proof, stable,\footnote{Stability requires the exit to be voluntary; that is, for any preference profile the social choice function has the property that all members belonging to the final society want to stay (internal stability) and all members who do not belong to the final society do not want to belong (external stability).} and respects voters’ sovereignty on the set of candidates is the voting by committees that requires unanimity for the entrance of each candidate. The dynamic aspect of this decision is hidden in the general formulation of the mechanism as a social choice function. In Berga, Bergantiños, Massó, and Neme (2003b) we concentrate on particular mechanisms where the final society, consisting of the subset of elected candidates and the subset of members that decide to stay in the society, is the outcome of a two-stage game; hence, formulating explicitly the dynamics of the decision. First, members choose a subset of candidates by a given voting procedure. Second, and after knowing the set of elected candidates \(X\), members of the society decide whether to stay or exit the society. This model is strategically rich because a member, when evaluating the consequences of a vote for a particular candidate \(x\), has to take into account (not only whether or not he likes \(x\) but also) two simultaneous effects (and their ramifications) of \(x\) being chosen. First, the choice of \(x\) might be used by member \(i\) to get rid of member \(j\) if \(i\) does not like \(j\) and \(j\) does not like candidate \(x\) (similar and even more involved consequences of \(x\) being chosen may arise as well; for instance, \(i\) might like \(j\) but not \(j'\) who belongs to the society just because \(j\) is a member of it, but \(j'\) would leave it as soon as \(j\) exits it; i.e., \(i\) votes for \(x\) to get rid of \(j'\) by bringing about the exit of \(j\)). Second, support of candidate \(x\) might be used by member \(i\) to keep member \(j''\), who is ready to leave the society whenever candidate \(y\) is chosen (the chosen one if
does not vote for \( x \), because \( j'' \)'s membership is critical for \( i \)'s continued presence in the society (and further obvious effects). In this setting, we exhibited an example without any subgame perfect Nash equilibria in pure strategies and another one in which in all subgame perfect Nash equilibria in pure strategies at least one player uses a dominated strategy.

For societies whose members perceive the membership of all other members as being desirable (monotonic preferences) we were able to identify, for each subset of elected candidates \( X \), a reasonable and meaningful subset of members that leave the society (\( EA(X) \), the exit set after \( X \) is chosen). This set has desirable properties and it is identified by means of a recursive process that mimics the iterative elimination of dominated strategies.

Here, we add more structure to the problem. First, we assume that the voting procedure is voting by committees. Second, we suppose that member’s preferences are not only additive in the sense of Barberà, Sonnenschein, and Zhou (1991) but also have dichotomous bads. A candidate \( x \) is bad for member \( i \) if adding \( x \) to any society makes the society worse for \( i \). An additive preference of member \( i \) has dichotomous bads if each bad candidate (if any) is either extremely bad (his entrance makes the society to be, in any circumstance, undesirable for member \( i \)) or mildly bad (his entrance does not affect his exit decision). Our main result here is that, under this preference domain, the game induced by a voting by committees without vetoers has at least an undominated Nash equilibrium. Its existence is proven by the following recursive construction. At each stage, each member votes for his best subset of candidates given the set that has already been admitted in the previous stages and taking into account the exit it will induce. Given their votes (for the stage) the set of candidates joining the society (at this stage) is chosen according to the voting by committees. The process ends at the stage where no additional candidate would be admitted. From the overall set of candidates that each member has voted for along this process, we construct a strategy profile (a simultaneous vote for each member) that is an undominated Nash equilibrium of the induced game with exit.

Before finishing this introduction we want to point out that our model is not limited to the interpretation given so far; i.e., the choice of the composition of the final society. It can be also used to analyze the problem where a society has to define its formal and public positions on a set of issues. One can think of political parties or religious communities deciding on different issues like abortion, death penalty, health reform, and so on.
The paper is organized as follows. In Section 2 we introduce our basic framework. In Section 3 we define the game induced by a voting by committees and the exit set after a subset of candidates has been elected. In Section 4 we describe the domain of preferences with dichotomous bads and we obtain some properties of the exit set under this preference domain. An undominated Nash equilibrium is constructed in Section 5. In Section 6 we state and prove our main result. Three Appendices at the end of the paper contain the proofs of three propositions omitted in the text.

2 Preliminaries

Let $N = \{1, ..., n\}$ be the set of members of a society who must first choose a subset of new members among the finite set of candidates $K$. Then, knowing the elected candidates, each member decides to stay or to leave the society. Members in $N$ have preferences over $2^K \times 2^N$, the set of all possible final societies. Namely, a final society is a pair $[X, S] \in 2^K \times 2^N$ where $X$ is the set of elected candidates and $S$ is the set of members who stay in the society given that $X$ has been elected.\(^3\) To simplify notation we will often denote a final society $[X, S]$ by $X \cup S$.

The preferences of member $i \in N$ over $2^K \times 2^N$, denoted by $R_i$, is a complete, reflexive, and transitive binary relation. As usual, let $P_i$ and $I_i$ denote the strict and indifference preference relations induced by $R_i$, respectively. We suppose that each member's preferences $R_i$ satisfies the following five conditions:

(C1) **Strictness:** For all $X, X' \subset K$ and $S, S' \subset N$ such that $[X, S] \neq [X', S']$ and $i \in S \cap S'$, either $[X, S] P_i [X', S']$ or $[X', S'] P_i [X, S]$.

(C2) **Indifference:** For all $X \subset K$ and $S \subset N$, $i /\notin S$ if and only if $[X, S] I_i [X, \emptyset]$. Moreover, for all $X, X' \in 2^K$, $[X, \emptyset] I_i [X', \emptyset]$.

(C3) **Loneliness:** $[\emptyset, \{i\}] P_i [\emptyset, \emptyset]$.

(C4) **Monotonicity:** For all $X \subset K$ and all $S \subset S' \subset N$ such that $i \in S$, $[X, S'] P_i [X, S]$.

(C5) **Additivity:** There exists $u_i : N \cup K \cup \emptyset \to \mathbb{R}$ such that $u_i (\emptyset) = 0$ and for all $S, S' \subset N$ and $X, X' \subset K$

\(^3\)When considering $K$ as the set of issues that the society has to decide upon, the interpretation of a final society is the subset of approved issues and the subset of members that remain in the society.
\[ [X, S] P_i [X', S'] \] if and only if \[
\begin{cases}
\sum_{j \in X \cup S} u_i(j) > \sum_{j \in X \cup S'} u_i(j) & \text{when } i \in S \cap S', \\
\sum_{j \in X \cup S} u_i(j) > 0 & \text{when } i \in S \text{ and } i \not\in S'.
\end{cases}
\]

**Strictness** means that member \( i \)'s preference relation over final societies containing himself is strict. **Indifference** says that if member \( i \) is not in the society he is indifferent about who belongs to it. **Loneliness** says that member \( i \) finds specific benefits to being the only member of the society. **Monotonicity** means that members consider the exit of other members undesirable, independently of the elected candidates. Notice that monotonicity does not impose any condition when comparing two final societies with different elected candidates. In particular, monotonicity admits the possibility that member \( i \) prefers to belong to a smaller society. **Additivity** means that members’ preferences are additively representable by utility functions.

Note that under monotonicity, loneliness implies *non-initial exit*; that is, \([\emptyset, N] P_i [\emptyset, \emptyset] \). However, the converse is not true. Notice also that under additivity, the strictness condition implies that \( u_i(x) \neq 0 \) for all \( x \in K \cup N \setminus \{i\} \). Then, by loneliness, \( u_i(i) > 0 \) for all \( i \in N \) and by monotonicity, \( u_i(j) > 0 \) for all \( j \in N \setminus \{i\} \). Moreover, under additivity the set of candidates can be partitioned into two disjoint sets. We say that candidate \( x \) is *good* for member \( i \) according to \( R_i \) whenever \( u_i(x) > 0 \); otherwise, we say that candidate \( x \) is *bad* for member \( i \) according to \( R_i \). Denote by \( G(R_i) \) and \( B(R_i) \) the set of good and bad candidates for \( i \) according to \( R_i \), respectively. We denote by \( R_i \) the set of member \( i \)'s preferences satisfying conditions (C1)-(C5) and by \( \mathcal{R} \) the Cartesian product \( \mathcal{R}_1 \times \ldots \times \mathcal{R}_n \).

### 3 Voting by Committees with Exit

In this paper we depart from Berga, Bergantiños, Massó, and Neme (2003b) and we define the following two-stage game. First, members choose a subset of candidates with a given voting by committees. Second, and after knowing the elected candidates, members decide whether to stay or exit the society. In this setting we are interested in identifying a meaningful undominated Nash equilibrium in pure strategies of this two-stage game.
### 3.1 Voting by Committees

Following Barberà, Sonnenschein, and Zhou (1991), a voting by committees is defined by a collection of families of winning coalitions (committees), one for each candidate, $W = (W_x)_{x \in K}$. Members vote for a subset of candidates. To be elected, a candidate must get the vote of all members of some coalition among those that are winning for that candidate. Formally, a *committee for* $x \in K$, denoted by $W_x$, is a non-empty family of non-empty coalitions of $N$ satisfying coalition monotonicity ($S \in W_x$ and $S \subset T$ imply $T \in W_x$). Given a committee $W_x$, its set of minimal winning coalitions is $W_x^m \equiv \{ S \in W_x \mid T \notin W_x \text{ for all } T \subset S \}$. Then, a voting procedure $v : (2^K)^N \to 2^K$ is *voting by committees* if there exists $W = (W_x)_{x \in K}$ such that for all $V = (V_1, ..., V_n) \in (2^K)^N$ and all $x \in K$,

$$x \in v(V) \iff \{ i \in N \mid x \in V_i \} \in W_x.$$

We say that $v$ has no *vetoers* if the corresponding committees $W = (W_x)_{x \in K}$ have the property that for all $x \in K$ and all $i \in N$ there exists $S \in W_x^m$ such that $i \notin S$. We say that member $i$ is a *dummy* for candidate $x$ according to $v$ if there does not exist $S \in W_x^m$ such that $i \in S$. Given an integer $1 \leq q \leq n$, a voting by committees $v$ is *voting by quota* $q$ if for all $V = (V_1, ..., V_n) \in (2^K)^N$ and $x \in K$,

$$x \in v(V) \text{ if and only if } \# \{ i \in N \mid x \in V_i \} \geq q,$$

where $\#$ stands for the cardinality of a set.

Barberà, Sonnenschein, and Zhou (1991) show that without exit, voting by committees constitute the full class of strategy-proof and onto social choice functions on the domain of both additive and separable preferences over all subsets of candidates. Berga, Bergantiños, Massó, and Neme (2003a) show that social choice functions that are strategy-proof, stable, and respect voters’ sovereignty on the set of candidates must be voting by committees and must satisfy the extreme condition that each member is a vetoer of all candidates. Hence, voting by quota $n$ is the unique strategy-proof and stable social choice function that respects voters’ sovereignty on the set of candidates.

### 3.2 Exit

Assume that the set of candidates $X \in 2^K$ has already been elected and all members know that. Now, each member has to decide whether or not to continue in the
society. But often, societies do not clearly specify the rules under which this exit takes place. Therefore, and to avoid to go into the specific details of these exit decisions (the order in which members have to decide as well as their information about the others’ decisions), we recursively define (following Berga, Bergantíños, Massó, and Neme, 2003b) the set of members leaving the society after $X$ is chosen.

Define first the set $EA^1(X)$ as the subset of members that unambiguously want to leave the society as the consequence of $X$ being chosen; that is, $EA^1(X) = \{ i \in N \mid [X, N \setminus \{i\}] P_i [X, N] \}$, or equivalently, $\{ i \in N \mid [X, \emptyset] P_i [X, N] \}$. Let $t \geq 1$ and assume $EA^t(X)$ has been defined for all $t'$ such that $1 \leq t' \leq t$. Then,

$$EA^{t+1}(X) = \left\{ i \in N \setminus \left( \bigcup_{t'=1}^{t} EA^{t'}(X) \right) \mid [X, \emptyset] P_i \left[ X, N \setminus \left( \bigcup_{t'=1}^{t} EA^{t'}(X) \right) \right] \right\}.$$ 

Let $t_X$ be either equal to 1 if $EA^1(X) = \emptyset$ or else be the smallest positive integer satisfying the property that $EA^{t_X}(X) \neq \emptyset$ but $EA^{t_X+1}(X) = \emptyset$. Then, define the exit set after $X$ is chosen as $EA(X) = \bigcup_{t=1}^{t_X} EA^t(X)$ and the exit function as $EA : 2^K \to 2^N$.

Observe that this set only depends on the preference profile $R$. Motivation and some of its properties can be found in Berga, Bergantíños, Massó, and Neme (2003b). In particular, $EA(X)$ is the set of members leaving the society after $X$ is chosen if exit is sequential (and members play according to the unique subgame perfect Nash equilibrium of the subgame starting at $X$); moreover, this set is independent of the ordering in which members decide (sequentially) whether to stay or to exit. The set $EA(X)$ also coincides with the set of members leaving the society if exit is simultaneous and members eliminate iteratively dominated strategies.

### 3.3 The Game

Fix a preference profile $R \in \mathcal{R}^N$. Given any voting by committees $v$ and the exit function $EA : 2^K \to 2^N$, we can model our two-stage game as the normal form game $\left( N, (2^K)^N, v, R \right)$. Given a strategy profile $V = (V_1, ..., V_n) \in (2^K)^N$ the final society is $v(V) \cup [N \setminus EA(v(V))]$. Since $N$ and $K$ are fixed we denote this game, given $v$ and $R$, by $\Gamma(v, R)$. To simplify notation, given a subset of candidates $X \subset K$ we use the notation $f(X)$ to express the final society when the set of candidates $X$ enter the society and the exit is given by $EA(X)$; i.e., $f(X) = X \cup (N \setminus EA(X))$. 


In addition and abusing notation, define for each $V \in (2^K)^N$,

$$
u_i(V) = \begin{cases} 
\sum_{j \in f(v(V))} u_i(j) & \text{if } i \not\in EA(v(V)) \\
0 & \text{if } i \in EA(v(V)).
\end{cases}
$$

Observe that a Nash equilibrium $V^*$ of $\Gamma(v, R)$ implicitly assumes that members, through $(EA(X))_{X \in 2^K}$, have a minimal rational behavior in all subgames starting at any $X$ (subgame perfection, for instance, if exit is sequential).

Berga, Bergantiños, Massó, and Neme (2003b) shows that the set of undominated Nash equilibria of $\Gamma(v, R)$ might be empty. To see that, consider the following example.

**Example 1** Let $N = \{1, 2, 3, 4\}$ be a society and let $K = \{x, y\}$ be the set of candidates. Assume that $v$ is voting by quota 1. Consider the additive preference profile $R \in \mathcal{R}^N$ represented by the utility functions $u_i : N \cup K \cup \emptyset \to \mathbb{R}$ given by the following table

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>100</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>100</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$x$</td>
<td>2</td>
<td>$-1$</td>
<td>$-5$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$y$</td>
<td>$-1$</td>
<td>2</td>
<td>$-5$</td>
<td>$-3$</td>
</tr>
</tbody>
</table>

For member 1, $\{y\}$ is dominated by $\emptyset$ and $\{x, y\}$ is dominated by $\{x\}$. For member 2, $\{x\}$ is dominated by $\emptyset$ and $\{x, y\}$ is dominated by $\{y\}$. For members 3 and 4, $\{x\}$, $\{y\}$, and $\{x, y\}$ are dominated by $\emptyset$. Therefore, the undominated strategies are $\{x\}$ and $\emptyset$ for member 1; $\{y\}$ and $\emptyset$ for member 2; $\emptyset$ for member 3; and $\emptyset$ for member 4. It is easy to check that the set of Nash equilibria of $\Gamma(v, R)$ is equal to

$$\{(V_1, V_2, V_3, V_4) \mid \# \{i \in N \mid x \in V_i\} \geq 2 \text{ and } \# \{i \in N \mid y \in V_i\} \geq 2\}.$$

Therefore, the set of undominated Nash equilibria of $\Gamma(v, R)$ is empty. \hfill $\square$

In the remainder of the paper we show that the game $\Gamma(v, R)$ has undominated Nash equilibria whenever the preference profile $R$ satisfies the property of having

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4We use the same notation $(u_i)$ for the utility function of member $i$ in the game $\Gamma(v, R)$ and the function representing the additive preference of member $i$. The reader will not be confused because from the context it will be clear which one of the two usages we are referring to.
dichotomous bads. We will do it by constructing a particular and meaningful voting strategy profile.

4 Dichotomous Bads

There are societies whose members clearly distinguish among bad candidates taking into account its “wickedness”. Think, as a first example, on the problem of a political party which has to decide, among other issues, its list of candidates of an election. Let $x$ be a potential candidate. Imagine that militant $i$ highly dislikes $x$ in such a way that $i$ will disaffiliate from the party if $x$ is chosen even in the best situation where all the other candidates in the list are good for $i$. In this case, we say that $x$ is an extremely bad candidate for member $i$, who will quit its militancy for sure if $x$ is selected. A second example could be related to the establishment of issues in the agenda of a political party: abortion, death penalty, tax reform, military intervention in another country, etc. Militant $i$ may also dislike another approved issue $y$ concerning for example public transportation policy. However, while the approval of a military intervention in a foreign country would trigger his exit, the approval of $y$ is never decisive in his exit. In such a case we say that $y$ is a mildly bad item for $i$. Formally, for member $i$’s preference $R_i \in \mathcal{R}_i$, we define the subsets of extremely bad, $B^{-\infty}(R_i)$, and mildly bad candidates, $B^{-\varepsilon}(R_i)$, as follows:

$$B^{-\infty}(R_i) \equiv \{ x \in B(R_i) \mid [G(R_i) \cup \{x\}, \emptyset] P_i [G(R_i) \cup \{x\} \cup \{x\}, N] \}$$

$$B^{-\varepsilon}(R_i) = \left\{ x \in B(R_i) \mid \forall X \in 2^K \text{ such that } [X, \{i\}] P_i [X, \emptyset], \text{ then } [X \cup \{x\}, \{i\}] P_i [X \cup \{x\}, \emptyset] \right\}.$$

The presence of an extremely bad candidate makes the society unacceptable for member $i$ in her most favorable scenario (with all the members and good candidates inside). Observe that for all $R_i \in \mathcal{R}_i$, $B^{-\infty}(R_i) \cap B^{-\varepsilon}(R_i) = \emptyset$.

**Remark 1** The society formed by member $i$ and her set of mildly bad candidates is, by its definition, acceptable for member $i$, that is, $[B^{-\varepsilon}(R_i), \{i\}] P_i [B^{-\varepsilon}(R_i), \emptyset]$.

We say that a preference relation partitioning bad candidates between these two types has dichotomous bads.

**Definition 1** A preference relation $R_i \in \mathcal{R}_i$ has dichotomous bads if $B(R_i) = B^{-\infty}(R_i) \cup B^{-\varepsilon}(R_i)$. 

9
Let $D_i \subset R_i$ be the subset of member $i$’s preferences with dichotomous bads and let $D$ denote the Cartesian product $D_1 \times \cdots \times D_n$. Observe that in Example 1, $R_4 \in R_4$ does not have dichotomous bads. $B^{-\infty} (R_4) = \emptyset$ since $\{x\}, N \cup \{x\}, \emptyset$ and $\{y\}, N \cup \{y\}$, which means that $B^{-\varepsilon} (R_4) \neq \{x, y\}$. In fact, $B^{-\varepsilon} (R_4) = \emptyset$. Next proposition characterizes $B^{-\infty} (R_i)$ and $B^{-\varepsilon} (R_i)$ in terms of the exit function $EA : 2^K \rightarrow 2^N$. It says that the entrance of an extremely bad candidate always produce the exit of member $i$ whereas the entrance of a mildly bad candidate does not affect the exit of $i$.

**Proposition 1** Let $i \in N$ and $R_i \in D_i$. Then,

- $B^{-\infty} (R_i) = \{x \in B (R_i) \mid i \in EA (X) \text{ whenever } x \in X \subset K\}$ and
- $B^{-\varepsilon} (R_i) = \{x \in B (R_i) \mid \text{for all } X \in 2^K, [i \in EA (X \cup \{x\}) \Leftrightarrow i \in EA (X)]\}$.

**Proof** See Appendix 1.

We now establish some useful properties of the exit function $EA : 2^K \rightarrow 2^N$ for the domain of preferences with dichotomous bads $D$.

**Proposition 2** Let $R = (R_1, \ldots, R_n) \in D$. Then, the following properties hold:

1. $EA (A) = \bigcup_{x \in A} EA (x)$ for all $A \subset K$.
2. Assume $A, B, C \subset K$ are such that $A \subset B$ and $B \cap C = \emptyset$. Then, $EA (B \cup C) \setminus EA (B) \subset EA (A \cup C) \setminus EA (A)$.
3. Assume $A, B, C \subset K$ are such that $A \subset B$ and $B \cap C = \emptyset$. Then,

$$\sum_{j \in f(B \cup C)} u_i (j) - \sum_{j \in f(B)} u_i (j) \geq \sum_{j \in f(A \cup C)} u_i (j) - \sum_{j \in f(A)} u_i (j).$$

**Proof** See Appendix 2.

Property (2.2) says that the exit produced by the additional entrance of new candidates ($C$) is larger the smaller is the set of elected candidates. Property (2.3) says that there are increasing returns to scale in the sense that the larger the set of elected candidates is the larger is the interest of members to accept new subsets of candidates.

## 5 An undominated Nash equilibrium

Let $W = (W_x)_{x \in K}$ be the set of families of winning coalitions defining the voting by committees $v$ and let $R = (R_1, \ldots, R_n) \in D$. To construct an undominated
Nash equilibrium of $\Gamma(v,R)$ we first consider the following process which may be understood as if each member would vote for candidates in successive stages.

- **Stage 1**: For all $i \in N$ define the set $\bar{V}_i^1$ as the best subset of candidates that member $i$ would like to admit taking into account the exit it would induce. Formally,

$$\bar{V}_i^1 = \{ X \in 2^K \mid f(X) P_i f(X') \text{ for all } X' \subset K \text{ such that } X' \neq X \}.$$  

By strictness and $EA(\emptyset) = \emptyset$, $\bar{V}_i^1$ is well defined and it contains a unique subset (possibly the empty set). Moreover,

$$f(\bar{V}_i^1) R_i [\emptyset, N] P_i [\bar{V}_i^1, \emptyset].$$

Therefore, $i \notin EA(\bar{V}_i^1)$. Set

$$V_i^1 = \bar{V}_i^1,$$

$$\bar{V}^1 = \{ x \in K \mid \{ i \in N \mid x \in \bar{V}_i^1 \} \in W_x \}, \text{ and }$$

$$V^1 = \bar{V}^1.$$  

Notice that $\bar{V}^1 = v((\bar{V}_i^1)_{i \in N})$; i.e., $\bar{V}^1$ is the set of elected candidates when members vote $(\bar{V}_i^1)_{i \in N}$.

- **Stage t+1**: Assume that $\bar{V}_i^r, \bar{V}^r, V_i^r, V^r$ have been defined for all $r \leq t$ and all $i \in N$ and $f(V^t) P_i [V^t, \emptyset]$ when $i \notin EA(V^t)$. We will define $\bar{V}_i^{t+1}$ and $V_i^{t+1}$ for all $i \in N$, and $V^{t+1}$ and $V^t$.

If $i \in EA(V^t)$, by property (2.1), there exists $x \in V^t \cap B^\infty(R_i)$ such that $i \in EA(x)$. Therefore, $i \in EA(V^t \cup X)$ for all $X \subset K \setminus V^t$. In this case we take $\bar{V}_i^{t+1} = \bar{V}_i^t$.

If $i \notin EA(V^t)$ then,

$$\bar{V}_i^{t+1} = \{ X \subset K \setminus V^t \mid f(V^t \cup X) P_i f(V^t \cup X') \text{ for all } X' \subset K \setminus V^t \text{ s.t. } X' \neq X \}.$$  

By strictness, $\bar{V}_i^{t+1}$ is well defined and it contains a unique subset (possibly the empty set). Moreover,

$$f(V^t \cup \bar{V}_i^{t+1}) R_i f(V^t) P_i [V^t, \emptyset].$$
Therefore, \( i \notin EA (V^t \cup \tilde{V}^{t+1}) \). Given \( V^t \), the set \( \tilde{V}^{t+1}_i \) is the best subset of candidates that member \( i \) would like to admit, once the set \( V^t \) has already been elected, taking into account the exit it would induce. Set

\[
V^{t+1}_i = \bigcup_{r=1}^{t+1} \tilde{V}^r_i,
\]

\[
\tilde{V}^{t+1} = \{ x \in K \setminus V^t \mid \{ i \in N \mid x \in \tilde{V}^{t+1}_i \} \in W_x \},
\]

and

\[
V^{t+1} = \bigcup_{r=1}^{t+1} V^r.
\]

The set \( V^{t+1}_i \) represents the candidates voted by \( i \) in stages 1, ..., \( t+1 \) while \( \tilde{V}^{t+1} \) represents the candidates joining the society when members vote \( (\tilde{V}^{t+1}_i)_{i \in N} \). Finally, \( V^{t+1} \) represents the candidates joining the society in stages 1, ..., \( t+1 \).

This process ends when there exists \( T \) satisfying \( \tilde{V}^{T-1} \neq \emptyset \) and \( \tilde{V}^T = \emptyset \). The following example illustrates this construction.

**Example 2** Let \( N = \{1, 2, 3\} \) be a society and let \( K = \{x, y\} \) be the set of candidates. Assume that \( v \) is voting by quota 1. Consider the additive preference profile \( R = (R_1, R_2, R_3) \in \mathcal{R} \) represented by the following utility functions:

<table>
<thead>
<tr>
<th></th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x )</td>
<td>2</td>
<td>5</td>
<td>-5</td>
</tr>
<tr>
<td>( y )</td>
<td>2</td>
<td>-1</td>
<td>-5</td>
</tr>
</tbody>
</table>

Note that \( R \in \mathcal{D} \). Moreover, \( B^{-\infty} (R_1) = B^{-\varepsilon} (R_1) = \emptyset \), \( B^{-\infty} (R_2) = \emptyset \), \( B^{-\varepsilon} (R_2) = \{y\} \), \( B^{-\infty} (R_3) = \{x, y\} \), and \( B^{-\varepsilon} (R_3) = \emptyset \). We now compute \( \tilde{V}^t_i \) for all \( i \) and \( t \).

<table>
<thead>
<tr>
<th>( t \setminus i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V^1_i )</td>
<td>\emptyset</td>
<td>( x )</td>
<td>\emptyset</td>
</tr>
<tr>
<td>( V^2_i )</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
<tr>
<td>( V^3_i )</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>

Then, \( T = 3 \), \( V^1_i = \{y\} \), \( V^2_i = \{x\} \), and \( V^3_i = \emptyset \). Observe that initially member 1 is not interested in the entrance of any of the two candidates since they produce the exit of member 3. But, once \( x \) is elected (member 2 likes him very much) and thus, member 3 exits, then member 1 wants \( y \) to be elected. Therefore, along the process, member 1 has only voted for \( y \) which is dominated by voting for both \( x \) and \( y \). \( \square \)
To avoid this, and since our objective is to identify an undominated Nash equilibrium, we modify the process by adding to each \( V_i^T \) the best subset \( H_i \) (taking into account its effects on the exit of other members) of the set \( A_i \) of good candidates that \( i \) has not voted for along the process but they joined the society before \( i \) exits. Formally, for each \( i \in N \),

\[
A_i = \begin{cases} 
  G(R_i) \cap (V_i^{T-1} \setminus V_i^T) & \text{if } i \notin EA(V^T) \\
  G(R_i) \cap (V_i^{t-1} \setminus V_i^T) & \text{if } i \notin EA(V^t) \text{ and } i \in EA(V^t) \text{ for some } t
\end{cases}
\]

and

\[H_i = \{ X \subset A_i \mid f(V_i^T \cup X) P_i f(V_i^T \cup X') \text{ for all } X' \subset A_i \text{ such that } X' \neq X \} .\]

We now prove that \( H_i \) is well defined. By convention we set \( V^0 = \emptyset \). We first prove that \( i \notin EA(V_i^T) \) considering two cases:

- Assume that \( i \in EA(V_i^T) \). Then, there exists \( h \in \{0, ..., T-1\} \) such that \( i \notin EA(V^h) \) but \( i \in EA(V^{h+1}) \). By the definition of the process \( \tilde{V}_i^T = \tilde{V}_i^h \) for all \( t = h + 1, ..., T \) and \( i \notin EA(V^t \cup \tilde{V}_i^{t+1}) \) for all \( t = 0, ..., h \). Then, by property (2.1), \( i \notin EA(\tilde{V}_i^t) \) for all \( t = 1, ..., h + 1 \). Since \( V_i^T = \bigcup_{t=1}^{h+1} \tilde{V}_i^t \), by property (2.1), \( i \notin EA(V_i^T) \).

- Assume now that \( i \notin EA(V_i^T) \). By the definition of the process \( i \notin EA(V^t \cup \tilde{V}_i^{t+1}) \) for all \( t = 0, ..., T-1 \). Then, by property (2.1), \( i \notin EA(\tilde{V}_i^t) \) for all \( t = 1, ..., T \). Since \( V_i^T = \bigcup_{t=1}^{h+1} \tilde{V}_i^t \), by property (2.1), \( i \notin EA(V_i^T) \).

Notice that, by definition of \( EA, f(V_i^T) P_i [V_i^T, \emptyset] \) because \( i \notin EA(V_i^T) \). By strictness we conclude that \( H_i \) is well defined, it is a singleton, and

\[ f(V_i^T \cup H_i) R_i f(V_i^T) P_i [V_i^T, \emptyset] . \]

Using the process defined above, we now define the vote of each member \( i \in N \) by

\[ V_i = V_i^T \cup H_i . \]

**Remark 2** By definition of \( EA, f(V_i) P_i [V_i, \emptyset] \) implies that \( i \notin EA(V_i) \) for all \( i \in N \).

Next proposition states two properties of the procedure defining \( V = (V_i)_{i \in N} \). First, if member \( i \) votes for candidate \( x \) in stage \( t \), \( i \) will vote for \( x \) in any later stage \( t' > t \), whenever \( x \) has not been elected yet. Second, when members vote
the set of elected candidates coincides with the set of elected candidates when members vote \((V_i)_{i \in N}\).

**Proposition 3** Let \(R \in D\). Then,

1. For all \(i \in N\) and \(1 \leq t \leq T - 1\), if \(x \in \bar{V}_t^i\) then, \(x \in \bar{V}_{t'}^i\) for any \(t' > t\) whenever \(x \notin V'^{t-1}\).
2. \(v\left((V_i^T)_{i \in N}\right) = v((V_i)_{i \in N}) = V^T\).

**Proof** See Appendix 3.

Observe that, as a consequence of property (3.1), given \(i \in N\) and \(1 \leq t \leq t' \leq T\), \(\bar{V}_t^i \subset V^t \cup \bar{V}_{t'}^i\).

### 6 The main result

Theorem 1 below states that if the society selects the candidates to become new members by a voting by committees without vetoers \(v\) and members follow the exit procedure given by \(EA\), \((V_1, \ldots, V_n)\) is an undominated Nash equilibrium of \(\Gamma(v, R)\).

**Theorem 1** Assume \(R \in D\) and let \(v\) be a voting by committees without vetoers. Then, \((V_1, \ldots, V_n)\) is an undominated Nash equilibrium of \(\Gamma(v, R)\).

The proof of Theorem 1 follows from the following three Lemmata.

**Lemma 2** Let \(i \in N\) and \(X \subset K\). Then,

1. to vote for \(X \cap G(R_i)\) is at least as good as to vote for \(X\);
2. if there exists \(x \in X \cap B(R_i)\) such that \(i\) is not a dummy for \(x\) then, to vote for \(X\) is dominated by to vote for \(X \cap G(R_i)\).

**Proof** Assume that \(X' = X \cap G(R_i)\).

a) We prove that given \(S'_i = X', S_i = X, S_j \subset K\) for all \(j \in N \setminus \{i\}\), then, \(u_i(S'_i, S_{-i}) \geq u_i(S_i, S_{-i})\). Take \(T' = v(S'_i, S_{-i})\) and \(T = v(S_i, S_{-i})\). By definition,

\[
u_i(S'_i, S_{-i}) = \begin{cases} \sum_{j \in N \setminus EA(T')} u_i(j) + \sum_{j \in T'} u_i(j) & \text{if } i \notin EA(T') \\ 0 & \text{otherwise} \end{cases}
\]

Since \(X' \subset X, T' \subset T\). By property (2.1), \(EA(T') \subset EA(T)\). Then, because preferences are monotonic,

\[\sum_{j \in N \setminus EA(T')} u_i(j) \geq \sum_{j \in N \setminus EA(T)} u_i(j)\]

Since \(X' = X \cap G(R_i), \sum_{j \in T'} u_i(j) \geq \sum_{j \in T} u_i(j)\). We consider three cases:
• $i \in EA(T') \subset EA(T)$. Then,

$$u_i (S_i', S_{-i}) = u_i (S_i, S_{-i}) = 0.$$

• $i \notin EA(T')$ but $i \in EA(T)$. Then,

$$u_i (S_i', S_{-i}) = \sum_{j \in N' \setminus EA(T')} u_i (j) + \sum_{j \in T'} u_i (j) \geq 0 = u_i (S_i, S_{-i}).$$

• $i \notin EA(T')$ and $i \notin EA(T)$. Then,

$$u_i (S_i', S_{-i}) = \sum_{j \in N' \setminus EA(T')} u_i (j) + \sum_{j \in T} u_i (j) \geq \sum_{j \in N' \setminus EA(T')} u_i (j) + \sum_{j \in T} u_i (j) = u_i (S_i, S_{-i}).$$

b) Take $X^* = \{x \in X \cap B (R_i) \mid i \text{ is not a dummy for } x \}$. Assume $X^* \neq \emptyset$. For any $x \in X^*$, let $W_x \in \mathcal{W}^m$ be such that $i \in W_x$. We now prove that there exists $S_{-i} = (S_j)_{j \in N \setminus \{i\}}$ satisfying $u_i (S_i', S_{-i}) > u_i (S_i, S_{-i})$. For each $j \in N \setminus \{i\}$ define $S_j = X' \cup \{x \in X^* \mid j \in W_x\}$. Then, $u (S_i', S_{-i}) = X'$ and $u (S_i, S_{-i}) = X' \cup X^* = Y$. Then,

$$\sum_{j \in X'} u_i (j) > \sum_{j \in Y} u_i (j).$$

By property (2.1), $EA(X') \subset EA(Y)$ and hence

$$\sum_{j \in N' \setminus EA(X')} u_i (j) \geq \sum_{j \in N' \setminus EA(Y)} u_i (j).$$

Then,

$$\sum_{j \in N' \setminus EA(X')} u_i (j) + \sum_{j \in X'} u_i (j) > \sum_{j \in N' \setminus EA(Y)} u_i (j) + \sum_{j \in Y} u_i (j).$$

Since $X' \subset G (R_i)$ we conclude that $i \notin EA(X')$. Then,

$$u_i (S_i', S_{-i}) = \sum_{j \in N' \setminus EA(X')} u_i (j) + \sum_{j \in X'} u_i (j) > 0.$$

We consider two cases.

• $i \in EA(Y)$. Then, $u_i (S_i', S_{-i}) > u_i (S_i, S_{-i})$ because $u_i (S_i, S_{-i}) = 0$. 

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\* \(i \notin \text{EA}(Y)\). Then, \(u_i(S'_i, S_{-i}) > u_i(S_i, S_{-i})\) because

\[
u_i(S_i, S_{-i}) = \sum_{j \in N \setminus \text{EA}(Y)} u_i(j) + \sum_{j \in Y} u_i(j).
\]

\[\blacksquare\]

**Lemma 3**  For each \(i \in N\), the strategy \(V_i\) is undominated.

**Proof** Assume that \(V_i\) is dominated by \(V'_i\) and \(V_i \neq V'_i\). Then, for each \(S = (S_1, ..., S_n) \in (2^k)^N\), \(u_i(V'_i, S_{-i}) \geq u_i(V_i, S_{-i})\). By Remark 2, \(i \notin \text{EA}(V_i)\). Since \(V_i\) is dominated by \(V'_i\) we conclude that \(i \notin \text{EA}(V'_i)\). By Lemma 2 a), we can assume that \(V'_i \cap B(R_i) = \emptyset\). We will get a contradiction by proving that \(V'_i = V_i\). We first prove that \(V_i \subset V'_i\) and later that \(V'_i \cap (K \setminus V_i) = \emptyset\).

1. \(V_i \subset V'_i\). We can assume, without loss of generality, that if \(i\) is a dummy for \(x\) and \(x \in V_i\) then \(x \in V'_i\). If not, take

\[V''_i = V'_i \cup \{x \in V_i \text{ such that } i \text{ is a dummy for } x\}\]

and proceed with \(V''_i\) instead of \(V'_i\). Notice that \(V''_i\) and \(V'_i\) are payoff equivalent for \(i\).

We first prove that \(V''_i \subset V'_i\) by induction on \(t\). We start by proving that \(V''_i \subset V'_i\). Suppose not. Then, \(Q = V''_i \setminus V'_i \neq \emptyset\). Given \(x \in Q\) let \(W_x \in W_x^m\) be such that \(i \in W_x\). We know that \(W_x\) exists because \(i\) is not a dummy for \(x\). For each \(j \in N \setminus \{i\}\) define \(S_j = V'_i \cup \{x \in Q \mid j \in W_x\}\). Of course, \(v(V'_i, S_{-i}) = V'_i\). Moreover, \(V''_i \subset v(V_i, S_{-i})\) because \(v\) has no vetoers. Since \(Q \subset V''_i \subset V_i\) it is easy to conclude that \(v(V_i, S_{-i}) = V'_i \cup Q\). Since \(i \notin \text{EA}(V_i)\) and \(i \notin \text{EA}(V'_i)\) we conclude, by property (2.1), that \(i \notin \text{EA}(V'_i \cup V_i)\) and \(i \notin \text{EA}(V'_i \cup Q)\).

Then,

\[
u_i(V_i, S_{-i}) - u_i(V'_i, S_{-i}) = \sum_{j \in f(V'_i \cup Q)} u_i(j) - \sum_{j \in f(V'_i)} u_i(j).
\]

By definition of \(\tilde{V}_i\) (\(= V'_i\)) and since \(\emptyset \neq Q \subset G(R_i)\),

\[
\sum_{j \in f(V'_i)} u_i(j) - \sum_{j \in f(V'_i \setminus Q)} u_i(j) > 0.
\]

Given \(A = V'_i \setminus Q = V'_i \setminus V'_i \cap V'_i, B = V'_i, \) and \(C = Q\), by property (2.3),

\[
u_i(V_i, S_{-i}) - u_i(V'_i, S_{-i}) \geq \sum_{j \in f(V'_i)} u_i(j) - \sum_{j \in f(V'_i \setminus Q)} u_i(j) > 0.
\]

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which contradicts that \( V_i \) is dominated by \( V'_i \). Then, \( V_i^t \subset V'_i \).

*Induction hypothesis:* Assume that \( V_i^t \subset V'_i \).

We now prove that \( V_i^{t+1} \subset V'_i \). We assume that \( i \notin EA(V^t) \), otherwise the result is trivial because \( V^t = V'_i \). Since \( V_i^{t+1} = V_i^t \cup \bar{V_i}^{t+1} \) it is enough to prove that \( \bar{V_i}^{t+1} \subset V'_i \). Suppose not. Then, \( Q = \bar{V_i}^{t+1} \setminus V'_i \neq \emptyset \). Given \( x \in Q \) let \( W_x \in W_{x}^{m} \) be such that \( i \in W_x \). We know that \( W_x \) exists because \( i \) is not a dummy for \( x \). For each \( j \in N \setminus \{i\} \) define \( S_j = V'_i \cup V^t \cup \{x \mid j \in W_x\} \). Since \( v \) has no vetoers, \( V'_i \cup V^t \subset v(V_i, S_{-i}) \) and \( V'_i \cup V^t \subset v(V'_i, S_{-i}) \). Now it is easy to conclude that \( v(V'_i, S_{-i}) = V'_i \cup V^t \) and \( v(V_i, S_{-i}) = V'_i \cup V^t \cup Q \). Then,

\[
  u_i(V_i, S_{-i}) - u_i(V'_i, S_{-i}) = \sum_{j \in f(V^t \cup V'_i \cup Q)} u_i(j) - \sum_{j \in f(V^t \cup V'_i)} u_i(j)
\]

By definition of \( \bar{V_i}^{t+1} \),

\[
  \sum_{j \in f(V^t \cup \bar{V_i}^{t+1})} u_i(j) - \sum_{j \in f(\bar{V_i}^{t+1} \setminus Q)} u_i(j) > 0.
\]

Given \( A = (V^t \cup \bar{V_i}^{t+1}) \setminus Q = V^t \cup (V_i^{t+1} \setminus Q) \), \( B = V^t \cup V'_i \) and \( C = Q \), by property (2.3),

\[
  u_i(V_i, S_{-i}) - u_i(V'_i, S_{-i}) \geq \sum_{j \in f(V^t \cup \bar{V_i}^{t+1})} u_i(j) - \sum_{j \in f(\bar{V_i}^{t+1} \setminus Q)} u_i(j) > 0,
\]

which contradicts that \( V_i \) is dominated by \( V'_i \). Then, \( \bar{V_i}^{t+1} \subset V'_i \) and hence, \( V_i^{t+1} \subset V'_i \).

We have proved that \( V_i^T \subset V'_i \). We now prove that \( H_i \subset V'_i \). Suppose not, then \( Q = H_i \setminus V'_i \neq \emptyset \). Given \( x \in Q \) let \( W_x \in W_{x}^{m} \) be such that \( i \in W_x \). We know that \( W_x \) exists because \( i \) is not a dummy for \( x \). For each \( j \in N \setminus \{i\} \) define \( S_j = V'_i \cup \{x \mid j \in W_x\} \). Using arguments similar to those already used before we conclude that \( v(V'_i, S_{-i}) = V'_i \) and \( v(V_i, S_{-i}) = V'_i \cup Q \). Then,

\[
  u_i(V_i, S_{-i}) - u_i(V'_i, S_{-i}) = \sum_{j \in f(V'_i \cup Q)} u_i(j) - \sum_{j \in f(V'_i)} u_i(j)
\]

By definition of \( H_i \),

\[
  \sum_{j \in f(V^T \cup H_i)} u_i(j) - \sum_{j \in f(V^T \cup (H_i \setminus Q))} u_i(j) > 0.
\]

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Given $A = (V^T \cup H_i) \setminus Q = V^T \cup (H_i \setminus Q)$, $B = V'_i$, and $C = Q$, by property (2.3),

$$u_i (V_i, S_{-i}) - u_i (V'_i, S_{-i}) \geq \sum_{j \in f(V^T \cup H_i)} u_i (j) - \sum_{j \in f(V^T \cup (H_i \setminus Q))} u_i (j) > 0,$$

which contradicts that $V_i$ is dominated by $V'_i$. Then, $H_i \subset V'_i$. Since $V_i = V^T_i \cup H_i$ we conclude that $V_i \subset V'_i$.

2. $V'_i \cap (K \setminus V_i) = \emptyset$. Suppose not. We already know that $V'_i \cap B (R_i) = \emptyset$. We can assume without loss of generality that if $i$ is a dummy for $x$ and $x \notin V_i$ then $x \notin V'_i$. If not, take

$$V''_i = V'_i \setminus \{x \in V'_i \text{ such that } i \text{ is a dummy for } x\}$$

and proceed with $V''_i$ instead of $V'_i$. Notice that $V''_i$ and $V'_i$ are payoff equivalent for $i$. We consider two cases:

- $i \notin EA (V^T)$.

  - We first prove that $(V'_i \setminus V_i) \cap (G (R_i) \setminus A_i) = \emptyset$. Suppose not. Then, $Q = (V'_i \setminus V_i) \cap (G (R_i) \setminus A_i) \neq \emptyset$. Given $x \in Q$ let $W_x \in W^m$ be such that $i \in W_x$. We know that $W_x$ exists because $i$ is not a dummy for $x$. By the process defining $V$ (step $T$) we know that for any subset $Q' \subseteq K$ if $V^{T-1} \cap Q' = \emptyset$ and $Q' \neq \bar{V}_{i}^{T}$,

  $$\sum_{j \in f(V^{T-1} \cup \bar{V}_{i}^{T})} u_i (j) > \sum_{j \in f(V^{T-1} \cup Q')} u_i (j).$$

  For each $j \in N \setminus \{i\}$ define $S_j = V^{T-1} \cup \bar{V}_{i}^{T} \cup \{x \in Q \mid j \in W_x\}$. Using arguments similar to those already used before we conclude that $V^{T-1} \cup \bar{V}_{i}^{T} \cup Q \subseteq v (V'_i, S_{-i})$ and $v (V_i, S_{-i}) = V^{T-1} \cup \bar{V}_{i}^{T}$. Take $Q' = v (V'_i, S_{-i}) \setminus V^{T-1}$, then

  $$u_i (V_i, S_{-i}) = \sum_{j \in f(V^{T-1} \cup \bar{V}_{i}^{T})} u_i (j)$$

  and

  $$u_i (V'_i, S_{-i}) = \sum_{j \in f(V^{T-1} \cup Q')} u_i (j).$$

  Since $V^{T-1} \cap Q' = \emptyset$ and $Q' \supseteq \bar{V}_{i}^{T} \cup Q \supseteq \bar{V}_{i}^{T}$, $u_i (V_i, S_{-i}) > u_i (V'_i, S_{-i})$, which contradicts that $V_i$ is dominated by $V'_i$. 18
We now prove that \((V'_i \setminus V_i) \cap A_i = \emptyset\). Suppose not. Then, \(Q = (V'_i \setminus V_i) \cap A_i \neq \emptyset\). Given \(x \in Q\) let \(W_x \in \mathcal{W}_x^m\) be such that \(i \in W_x\). We know that \(W_x\) exists because \(i\) is not a dummy for \(x\). By definition of \(H_i\) we know that for all \(Q' \subset A_i\), \(Q' \neq H_i\),

\[
\sum_{j \in f((V'_i \cup H_i))} u_i(j) > \sum_{j \in f((V'_i \cup Q'))} u_i(j).
\]

For each \(j \in N \setminus \{i\}\) define \(S_j = V_i^T \cup H_i \cup \{x \in Q \mid j \in W_x\}\). Using arguments similar to those already used before we conclude that \(V_i^T \cup H_i \cup Q \subset v(V'_i, S_{-i})\) and \(v(V_i, S_{-i}) = V_i^T \cup H_i\). Since \((V'_i \setminus V_i) \cap (G(R_i) \setminus A_i) = \emptyset\) we conclude that \(v(V'_i, S_{-i}) = V_i^T \cup H_i \cup Q\). Take \(Q' = H_i \cup Q\), then

\[
u_i(V_i, S_{-i}) = \sum_{j \in f(V_i^T \cup H_i)} u_i(j)
\]

and

\[
u_i(V'_i, S_{-i}) = \sum_{j \in f(V_i^T \cup Q')} u_i(j).
\]

Since \(Q' \subset A_i\) and \(Q' \neq H_i\) \((Q \neq \emptyset)\) we conclude that \(u_i(V_i, S_{-i}) > u_i(V'_i, S_{-i})\), which contradicts that \(V_i\) is dominated by \(V'_i\).

\(\bullet\) \(i \in EA(V^T)\). Then, there exists \(t < T\) such that \(i \notin EA(V^{t-1})\), \(i \in EA(V^t)\), and \(V_i = V_i^T \cup H_i\). Using the same arguments that in the case \(i \notin EA(V^T)\) we obtain a contradiction.

\[\blacksquare\]

**Lemma 4** \((V_1, \ldots, V_n)\) is a Nash equilibrium of \(\Gamma(v, R)\).

**Proof** We prove that for all \(i \in N\), \(u_i(V_i, V_{-i}) \geq u_i(V'_i, V_{-i})\) for all \(V'_i \in 2^K\).

Assume that \(i \in EA(V^T)\). By property (2.1) there exists \(x \in V^T\) such that \(i \in EA(x)\). Because of the process defining \(V\) there exists \(t \leq T\) such that \(\{j \in N \setminus \{i\} \mid x \in V_j^T\} \in \mathcal{W}_x\) and hence \(\{j \in N \setminus \{i\} \mid x \in V_j\} \in \mathcal{W}_x\). Then, \(x \in v(V'_i, V_{-i})\), which means that \(i \in EA(V'_i, V_{-i})\). Therefore, \(u_i(V) = u_i(V'_i, V_{-i})\).

Assume now that \(i \notin EA(V^T)\). We prove it by contradiction. Suppose that \(u_i(V'_i, V_{-i}) > u_i(V_i, V_{-i})\). By property (3.2), \(v(V_i, V_{-i}) = V^T\) and hence, \(u_i(V_i, V_{-i}) = \sum_{j \in f(V^T)} u_i(j)\). Since \(u_i(V'_i, V_{-i}) > u_i(V_i, V_{-i})\) and \(i \notin EA(V^T)\), \(u_i(V'_i, V_{-i}) = \sum_{j \in f(V')} u_i(j)\), where \(V' = v(V'_i, V_{-i})\). We assume, without loss of generality, that \(V'_i \setminus (V_i \cup V') = \emptyset\). If not, take \(V''_i = V'_i \cap (V_i \cup V')\) and proceed with \(V''_i\) instead of \(V'_i\) because \(v(V''_i, V_{-i}) = v(V'_i, V_{-i})\). We proceed in two steps.
1. We prove that if \( Q = V'_t \cap (V' \setminus V^T) \neq \emptyset \) then \( V^*_t = V'_t \cap V_t \) satisfies

\[
\sum_{j \in f'(V')} u_i(j) > \sum_{j \in f(V^T)} u_i(j),
\]

where \( V^* = v(V^*_t, V_{-i}) \). Notice that \( V' = V^* \cup Q \), and \( V^* \subset V^T \). Since \( V^T = V^{T-1} \), \( Q \subset V' \), and \( Q \cap V^T = \emptyset \). Because of the definition of Stage \( T \) of the process, \( Q \cap V^T_t = \emptyset \). Then, by definition of \( V^T_t \),

\[
\sum_{j \in f(V^T_t)} u_i(j) > \sum_{j \in f(V^T \cup Q)} u_i(j).
\]

Given \( A = V^T, B = V^T \cup \bar{V}^T_t \), and \( C = Q \), by property (2.3),

\[
\sum_{j \in f(V^T \cup Q)} u_i(j) - \sum_{j \in f(V^T \cup \bar{V}^T_t)} u_i(j) \geq \sum_{j \in f(V^T \cup Q)} u_i(j) - \sum_{j \in f(V^T)} u_i(j).
\]

Then,

\[
\sum_{j \in f(V^T)} u_i(j) > \sum_{j \in f(V^T \cup Q)} u_i(j).
\]

We know that \( V^* \subset V^T \). If \( V^* = V^T \) we get a contradiction because \( V' = V^* \cup Q = V^T \cup Q \). Assume that \( V^* \neq V^T \). Given \( A = V^*, B = V^T \), and \( C = Q \), by property (2.3),

\[
\sum_{j \in f(V^T \cup Q)} u_i(j) - \sum_{j \in f(V^T)} u_i(j) \geq \sum_{j \in f(V^T \cup Q)} u_i(j) - \sum_{j \in f(V^*)} u_i(j).
\]

Then,

\[
\sum_{j \in f(V^*)} u_i(j) > \sum_{j \in f(V^T \cup Q)} u_i(j) > \sum_{j \in f(V^T)} u_i(j).
\]

As a consequence of this part we can assume, without loss of generality, that \( V'_t \subset V_t \). Then, \( V' \subset V^T \).

2. For each \( t = 1, \ldots, T \) we define \( S^t = \bar{V}^t \cap (K \setminus V') \). Assume that \( x \in S^t \). According with \( V_t \), candidate \( x \) is elected in stage \( t \) of the process. According with \( V'_t \), \( x \) is not elected. Then, member \( i \) votes for \( x \) in Stage \( t \) (\( x \in \bar{V}^t_t \)).

Notice that \( V' = \bigcup_{t=1}^{T} (\bar{V}^t \setminus S^t) \). We will get a contradiction by proving that

\[
\sum_{j \in f(V^T)} u_i(j) \geq \sum_{j \in f(\bar{V}^T)} u_i(j).
\]

For all \( t = 1, \ldots, T + 1 \) we define

\[
R^t = V^{t-1} \cup (\bigcup_{a=1}^{T} (\bar{V}^a \setminus S^a)) \).
\]
Observe that $R^1 = V'$, $R^{T+1} = V^T$ and $R_t^T + 1 = R_t^T \cup S_t^T$ for all $t = 1, \ldots, T - 1$ (by convenience we take $V^0 = \emptyset$). We proceed by induction. We first prove that $R_T^T$ satisfies
\begin{equation}
\sum_{j \in f(I_{R_t^T} + 1)} u_i(j) \geq \sum_{j \in f(I_{R_t^T})} u_i(j).
\end{equation}
Notice that $R_T^T = V_T^T - 1 \cup (\bar{V}_T^T \setminus S_T^T) = V_T^T$ because $\bar{V}_T^T = \emptyset$. Then, the result holds trivially. Assume now that for all $t = t^* + 1, \ldots, T$,
\begin{equation}
\sum_{j \in f(I_{R_t^T + 1})} u_i(j) \geq \sum_{j \in f(I_{R_t^T})} u_i(j).
\end{equation}
We now prove that
\begin{equation}
\sum_{j \in f(I_{R_t^T + 1})} u_i(j) - \sum_{j \in f(I_{R_t^*})} u_i(j) \geq 0.
\end{equation}

If $S_{t^*} = \emptyset$ the result holds trivially because $R_{t^*} + 1 = R_{t^*}$. Assume that $S_{t^*} \neq \emptyset$.

By definition of $\bar{V}_t^{t^*}$ we know that
\begin{equation}
\sum_{j \in f(V_t^{t^*} - 1 \cup \bar{V}_t^{t^*})} u_i(j) - \sum_{j \in f(V_t^{t^*} - 1 \cup (\bar{V}_t^{t^*} \setminus S_t^{t^*}))} u_i(j) > 0.
\end{equation}

Given $A = V_t^{t^* - 1} \cup (\bar{V}_t^{t^*} \setminus S_t^{t^*})$, $B = R_t^{t^*}$, and $C = S_t^{t^*}$, by property (2.3),
\begin{equation}
\sum_{j \in f(I_{R_t^T + 1})} u_i(j) - \sum_{j \in f(I_{R_t^*})} u_i(j) \geq \sum_{j \in f(V_t^{t^*} - 1 \cup \bar{V}_t^{t^*})} u_i(j) - \sum_{j \in f(V_t^{t^*} - 1 \cup (\bar{V}_t^{t^*} \setminus S_t^{t^*}))} u_i(j) \geq 0.
\end{equation}

Then, $\sum_{j \in f(V_t^T)} u_i(j) = \sum_{j \in f(I_{R_t^T + 1})} u_i(j) \geq \sum_{j \in f(I_{R_t^1})} u_i(j) = \sum_{j \in f(V_t^1)} u_i(j)$, which is the desired contradiction. \qed

Next example shows that there might be other undominated Nash equilibria different from $(V_1, \ldots, V_n)$.

**Example 3** Let $N = \{1, 2, 3\}$ be a society and let $K = \{x, y\}$ be the set of candidates. Assume that $v$ is voting by quota 1. Consider the preference profile $R = (R_1, R_2, R_3) \in D$ represented by the following utility functions:

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$y$</td>
<td>-1</td>
<td>2</td>
<td>-10</td>
</tr>
</tbody>
</table>

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In this case $V_1 = V_2 = V_3 = \emptyset$. Consider the strategy profile $V' = (\{x\}, \{y\}, \emptyset)$. It is straightforward to show that $V'$ is a Nash equilibrium of $\Gamma(v, R)$. Moreover, $\{x\}$ is undominated for member 1 because when member 2 votes $\{y\}$ the best reply of member 1 is to vote $\{x\}$. With similar arguments we can conclude that to vote $\{y\}$ is undominated for member 2. Of course, $\emptyset$ is undominated for member 3. Therefore, $V'(\neq V)$ is an undominated Nash equilibrium of $\Gamma(v, R)$. 

\section{References}


Appendix 1: Proof of Proposition 1

Before proving Proposition 1, we establish a useful Lemma. Observe that monotonicity and loneliness imply $EA(\emptyset) = \emptyset$.

**Lemma 1** Assume $i \in EA(X)$. Then, there exists $x \in X$ such that $x \in B^{-\infty}(R_i)$.

**Proof** Since $i \in EA(X) = \bigcup_{t=1}^{T} EA^t(X)$ we conclude that $i \in EA^{t_0}(X)$ for some $t_0 \leq T$ and hence,

$$[X, \emptyset] P_i [X, N \setminus \bigcup_{t=1}^{t_0-1} EA^t(X)].$$

Since preferences are monotonic,

$$[X, N \setminus \bigcup_{t=1}^{t_0-1} EA^t(X)] R_i [X, \{i\}].$$

By contradiction, assume that $X \cap B^{-\infty}(R_i) = \emptyset$. Since $R_i$ has dichotomous bads,

$$[X, \{i\}] R_i [B^{-\varepsilon}(R_i), \{i\}].$$

By indifference and transitivity of $R_i$,

$$[B^{-\varepsilon}(R_i), \emptyset] I_i [X, \emptyset] P_i [B^{-\varepsilon}(R_i), \{i\}].$$

But this is a contradiction since $[B^{-\varepsilon}(R_i), \{i\}] P_i [B^{-\varepsilon}(R_i), \emptyset]$ by Remark 1. ■

**Proof of Proposition 1** We need to prove the following four statements:

(a) $B^{-\infty}(R_i) \subset \{x \in B(R_i) \mid i \in EA(X) \text{ whenever } x \in X \subset K\}$.

Take $x \in B^{-\infty}(R_i)$ and $X \subset K$ such that $x \in X$. By indifference,

$$[X, \emptyset] I_i [G(R_i) \cup \{x\}, \emptyset].$$

Since $x \in B^{-\infty}(R_i)$,

$$[G(R_i) \cup \{x\}, \emptyset] P_i [G(R_i) \cup \{x\}, N].$$

By additivity,

$$[G(R_i) \cup \{x\}, N] R_i [X, N],$$

where $R_i = I_i$ if $X = G(R_i) \cup \{x\}$ and $R_i = P_i$ if $X \neq G(R_i) \cup \{x\}$, by strictness. By transitivity, $[X, \emptyset] P_i [X, N]$. Hence, $i \in EA^1(X) \subset EA(X)$.
Proof of Proposition 2

Let \( x \in E(A(X) \ \text{whenever} \ x \in X \subset K \subset B^{-\infty}(R_i) \).

Take \( X = \{x\}. \) Since \( i \in E(A(X) \), by Lemma 1, there exists \( y \in B^{-\infty}(R_i) \cap X. \) Then, \( x \in B^{-\infty}(R_i) \).

(c) \( B^{-\varepsilon}(R_i) \subset \{x \in B(R_i) \ | \ \text{for all} \ X \subset K, \ [i \in E(A(X \cup \{x\}) \iff i \in E(A(X))] \}. \)

Take \( x \in B^{-\varepsilon}(R_i) \) and \( X \subset K. \) We now prove that \( i \in E(A(X \cup \{x\}) \iff \text{for all} \ X \subset K, \ [i \in E(A(X \cup \{x\}) \iff i \in E(A(X))] \). Assume that \( i \in E(A(X) \). Then, by Lemma 1, there exists \( y \in X \) such that \( y \in B^{-\infty}(R_i) \). By (a), we conclude that \( i \in E(A(X \cup \{x\}) \iff i \in E(A(X)) \). Assume that \( i \in E(A(X \cup \{x\}) \). Then, by Lemma 1, there exists \( y \in X \cup \{x\} \) such that \( y \in B^{-\infty}(R_i) \). Since \( R_i \) has dichotomous bads and \( x \in B^{-\varepsilon}(R_i) \) we conclude that \( y \neq x. \) Then, \( y \in X \cap B^{-\infty}(R_i) \) and hence \( i \in E(A(X) \) because of (a).

(d) \( \{x \in B(R_i) \ | \ \text{for all} \ X \subset K, \ [i \in E(A(X \cup \{x\}) \iff i \in E(A(X))] \} \subset B^{-\varepsilon}(R_i) \).

Since \( R_i \) has dichotomous bads it is enough to prove that if \( x \in B^{-\infty}(R_i) \) then \( x \) does not satisfy that for all \( X \subset K, \ i \in E(A(X) \) if and only if \( i \in E(A(X \cup \{x\})) \). Assume \( X = \emptyset \). By loneliness and monotonicity, there is no initial exit; i.e., \( E(A(\emptyset) = \emptyset \) and hence \( i \notin E(A(\emptyset). \) But \( i \in E(A(\{x\}) \) because \( x \in B^{-\infty}(R_i). \)

Appendix 2: Proof of Proposition 2

Proof of Proposition 2 Let \( i \in E(A) \). By Lemma 1, there exists \( x \in A \cap B^{-\infty}(R_i) \). By part (a) in the proof of Proposition 1, \( i \in E(A(X) \) for every \( X \subset K \) such that \( x \in X \). Then, \( i \in E(A(x) \) and \( E(A(A) \subset \cup_{x \in A} E(A(x). \) Now, assume that \( i \in E(A(x) \) with \( x \in A \). Then, by Lemma 1, \( x \in B^{-\infty}(R_i) \) and hence, by Proposition 1, \( i \in E(A(A). \) This proves (2.1).

To prove properties (2.2) and (2.3) assume that \( A, B, C \subset K \) are such that \( A \subset B \) and \( B \cap C = \emptyset. \) By property (2.1) we have:

\[
E(A(B \cup C) \setminus E(A(B) = \bigcup_{x \in B \cup C} E(A(x) \setminus \bigcup_{x \in B} E(A(x) \quad (1)
\]

\[
= \bigcup_{x \in C} E(A(x) \setminus \bigcup_{x \in B} E(A(x)
\]

\[
= E(A(C) \setminus E(A(B),
\]

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and

\[ EA(A \cup C) \setminus EA(A) = \bigcup_{x \in A \cup C} EA(x) \setminus \bigcup_{x \in A} EA(x) \]
\[ = \bigcup_{x \in C} EA(x) \setminus \bigcup_{x \in A} EA(x) \]
\[ = EA(C) \setminus EA(A). \]  

Because \( EA(A) \subset EA(B) \), (1) and (2) imply that property (2.2) holds.

By definition of \( f \):

\[
\begin{align*}
    f(B \cup C) &= (B \cup C) \cup [N \setminus EA(B \cup C)] \\
    f(B) &= B \cup [N \setminus EA(B)] \\
    f(A \cup C) &= (A \cup C) \cup [N \setminus EA(A \cup C)] \\
    f(A) &= A \cup [N \setminus EA(A)].
\end{align*}
\]

Thus,

\[
\sum_{j \in f(B \cup C)} u_i(j) - \sum_{j \in f(B)} u_i(j) = \sum_{j \in B \cup C} u_i(j) + \sum_{j \in N \setminus EA(B \cup C)} u_i(j) - \sum_{j \in B} u_i(j) - \sum_{j \in N \setminus EA(B)} u_i(j).
\]

Because \( B \cap C = \emptyset \) and \( N \setminus EA(B) \subseteq N \setminus EA(B \cup C) \),

\[
\sum_{j \in f(B \cup C)} u_i(j) - \sum_{j \in f(B)} u_i(j) = \sum_{j \in C} u_i(j) - \sum_{j \in N \setminus EA(B \cup C) \setminus EA(B)} u_i(j).
\]

On the other hand,

\[
\sum_{j \in f(A \cup C)} u_i(j) - \sum_{j \in f(A)} u_i(j) = \sum_{j \in A \cup C} u_i(j) + \sum_{j \in N \setminus EA(A \cup C)} u_i(j) - \sum_{j \in A} u_i(j) - \sum_{j \in N \setminus EA(A)} u_i(j).
\]

Because \( A \cap C = \emptyset \) and \( N \setminus EA(A) \subseteq N \setminus EA(A \cup C) \),

\[
\sum_{j \in f(A \cup C)} u_i(j) - \sum_{j \in f(A)} u_i(j) = \sum_{j \in C} u_i(j) - \sum_{j \in N \setminus EA(A \cup C) \setminus EA(A)} u_i(j).
\]

By monotonicity and property (2.2),

\[
\sum_{j \in C} u_i(j) - \sum_{j \in EA(B \cup C) \setminus EA(B)} u_i(j) \geq \sum_{j \in C} u_i(j) - \sum_{j \in EA(A \cup C) \setminus EA(A)} u_i(j).
\]

Hence, property (2.3) holds. \( \blacksquare \)
Appendix 3: Proof of Proposition 3

Proof of Proposition 3  (3.1) If \( i \in EA(V^t) \) then, by the definition of the process, \( \bar{V}_i^{t'} = \tilde{V}_i^t \) for all \( t' > t \). Hence, \( x \in \bar{V}_i^{t'} \).

Assume now that \( i \notin EA(V^t) \) and let \( t' > t \). We consider two cases:

- We first assume that \( i \notin EA(V^{t-1}) \). Suppose that \( x \notin \bar{V}_i^{t'} \). Then, \( Q = V_i^t \setminus (V^{t-1} \cup \bar{V}_i^{t'}) \neq \emptyset \) because \( x \in Q \). Applying property (2.3) to \( A = V^{t-1} \cup (\bar{V}_i^t \setminus Q) \), \( B = V^{t-1} \cup \bar{V}_i^{t'} \), and \( C = Q \),

\[
\sum_{j \in f(V^{t-1} \cup \bar{V}_i^{t'} \cup Q)} u_i(j) - \sum_{j \in f(V^{t-1} \cup \bar{V}_i^{t'})} u_i(j) \geq \sum_{j \in f(V^{t-1} \cup \bar{V}_i^t)} u_i(j) - \sum_{j \in f(V^{t-1} \cup (\bar{V}_i^t \setminus Q))} u_i(j) > 0,
\]

where the last inequality comes from the definition of \( \bar{V}_i^t \). Then,

\[
f \left( V^{t-1} \cup \bar{V}_i^{t'} \cup Q \right) P_i f \left( V^{t-1} \cup \bar{V}_i^{t'} \right),
\]

which contradicts the definition of \( \bar{V}_i^{t'} \).

- We now assume that \( i \in EA(V^{t-1}) \). Let \( t^* > t \) be such that \( i \notin EA(V^{t^*}) \) but \( i \in EA(V^{t^*+1}) \). We know that \( t \leq t^* < t' \) and \( t' > t + 1 \). Because of the previous case we know that \( x \in \bar{V}_i^{t^*+1} \). By the definition of the process, \( \bar{V}_i^{t'} = \bar{V}_i^{t^*-1} = \ldots = \bar{V}_i^{t^*+1} \). Hence, \( x \in \bar{V}_i^{t'} \).

(3.2) We first prove that \( v \left( (V_i^T)_{i \in N} \right) = V^T \). Of course, \( V^T \subset v \left( (V_i^T)_{i \in N} \right) \).

We prove that \( v \left( (V_i^T)_{i \in N} \right) \subset V^T \). Suppose that \( x \in v \left( (V_i^T)_{i \in N} \right) \) but \( x \notin V^T \). Then, \( \{ i \in N \mid x \in V_i^T \} \in W_x \). If \( x \in V_i^T \) there exists \( 1 \leq t_i \leq T \) such that \( x \in \tilde{V}_i^{t_i} \), because \( V_i^T = \bigcup_{t=1}^{T} \tilde{V}_i^t \). By property (3.1), \( x \in \bar{V}_i^T \) because \( x \notin V_i^T \), \( V_i^T = \bigcup_{t=1}^{T} \tilde{V}_i^t \), and \( x \in \bar{V}_i^{t_i} \). Then, \( \{ i \in N \mid x \in V_i^T \} \in W_x \) and hence, \( x \in V^T \subset V^T \), which is a contradiction.

We now prove that \( v \left( (V_i^T)_{i \in N} \right) = v \left( (V_i)_{i \in N} \right) \). We know that \( V_i^T \subset V_i \) for all \( i \in N \). Then, \( v \left( (V_i^T)_{i \in N} \right) \subset v \left( (V_i)_{i \in N} \right) \). Suppose that \( x \in v \left( (V_i)_{i \in N} \right) \) but \( x \notin v \left( (V_i^T)_{i \in N} \right) \). Then, \( x \in H_i \) for some \( i \in N \) and hence, by definition of \( H_i \), \( x \in V^T \subset v \left( (V_i^T)_{i \in N} \right) \), which is a contradiction. \( \blacksquare \)