Working Paper Series

Demand bargaining and proportional payoffs in legislatures

María Montero and Juan J. Vidal-Puga
6-07
Demand bargaining and proportional payoffs in legislatures∗

Maria Montero†       Juan J. Vidal-Puga‡

July 28, 2007

Abstract

We study a legislative bargaining model in which the parties make payoff demands in decreasing order of voting weight. The unique subgame perfect equilibrium outcome is such that the minimal winning coalition of the parties that move first forms with payoffs proportional to the voting weights.

Key Words: Legislative bargaining; demand commitment; coalition formation; weighted majority games; minimal winning coalitions.

J.E.L. Classification Numbers: C72, C78, D72.

∗We would like to thank Massimo Morelli, Alex Possajennikov, Daniel Seidmann, participants in the Midlands Game Theory Workshop (December 2006) and seminar participants at Queen’s University Belfast for helpful comments. The second author acknowledges financial support from the Spanish Ministerio de Ciencia y Tecnología and FEDER through grant SEJ2005-07637-C02-01/ECON and from the Xunta de Galicia through grants PGIDIT06PXIC300184PN and PGIDT06PXIB362390.

†Corresponding author. School of Economics, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom. Phone + 44 115 9515468, fax +44 115 951 4159, e-mail: maria.montero@nottingham.ac.uk.

‡Research Group of Economic Analysis (RGEA) and Department of Statistics and Operations Research, University of Vigo. Facultad de Ciencias Sociales, Campus A Xunqueira, 36005 Pontevedra, Spain. Phone +34 986 802014, fax +34 986 813511, e-mail vidalpuga@uvigo.es.
1 Introduction

In a parliamentary democracy, many important decisions including government formation are the outcome of bargaining between the parties in Parliament. The most influential model of legislative bargaining is the closed rule model of Baron and Ferejohn [4], which is based on Rubinstein [25] and Binmore [7].\(^1\) In this model, a party is randomly recognized to propose a complete distribution of ministerial payoffs and the remaining parties vote on the proposal. The predictions of this model have some properties that may be perceived as drawbacks: the proposer has a large advantage (it receives more than half of the total payoff under simple majority), and there is a multiplicity of subgame perfect equilibria. In order to single out a unique prediction, the stationary equilibrium is selected. Stationary strategies are simple but by no means uncontroversial: a stationary strategy requires a party to always make the same proposal regardless of the history of the negotiations so far. Moreover, Norman [22] shows that sharp predictions using stationarity are only possible in the infinite horizon version of the model: in the finite horizon version there is a continuum of equilibria, all of them with history-independent strategies.

An alternative model of legislative bargaining by Morelli [20] is based not on complete proposals but on demands.\(^2\) Parties make individual demands

---

\(^1\)This model has led to many applications and extensions. Recent related papers include Banks and Duggan [3], Diermeier et al. [12] and Kalandrakis [15].

\(^2\)There have been several demand bargaining models in the literature. Binmore [6] presents a three-player model where demands are carried over to the next round and infinite plays are possible. Selten [27] presents a general but complex model, including random draws and costs of formulating a demand and of forming a coalition. Bennett and van Damme [5] study a simpler version in which each player selects the next one to move, and show that there may be a multiplicity of subgame perfect equilibria. Using a refinement, they select the proportional payoff division for apex games. Cardona-Coll and Mancera [9] analyze a legislative bargaining game in which parties are randomly chosen to make a demand which is then voted upon, and show that the surplus may be universally shared. Winter [32], Dasgupta and Chiu [11], and Vidal-Puga [28] use various demand
for ministerial payoffs and a coalition emerges between parties making compatible demands. The Head of State chooses the first mover, and the latter chooses the order in which the parties formulate demands. Because the first mover chooses the order of moves, it may be able to play the remaining parties off against each other and obtain the whole payoff, even though the rules of the game allow the other parties to exclude the first mover (see Montero and Vidal-Puga [19]).

In this paper we study a demand bargaining procedure in which the parties must move in decreasing order of voting weight. This procedure mirrors the assumption of Austen-Smith and Banks [2] that parties are asked to try to form a government in decreasing order of size. This is not a completely unnatural assumption: in some countries the largest party must be asked to form the government first, and in the absence of this rule the largest party is still selected disproportionately often.\footnote{The Greek constitution prescribes that the leader of the largest party must be chosen as the first formateur; if he fails, the leader of the second largest party is selected, to be followed by the leader of the third largest party if he too fails. Even if the constitution is silent on this matter, a convention may emerge (Laver and Schofield, [16], p. 210). For a quantitative analysis of formateur selection see table 1 in Warwick [30], table 3 in Diermeier and Merlo [13], and table 3 in Ansolabehere et al. [1].}

We analyze situations in which the distribution of seats in Parliament is well-behaved so that the associated majority game is constant-sum (no ties are possible) and homogeneous (even if not all minimal winning coalitions control the same number of seats, we can assign voting weights to the parties so that all minimal winning coalitions have the same total voting weight). Our main result is that the subgame perfect equilibrium outcome is unique, and equilibrium payoffs inside the coalition that forms are proportional to commitment procedures to implement the Shapley value in convex games. The procedure in [32] is similar to that in [5]. In both [11] and [28] players make sequential demands in a randomly predetermined order. In [11] any player may form a coalition with some players who already made their demands and end the game; in [28] the last mover is the only one able to form a coalition.
the homogeneous voting weights. The first mover has no disproportionate advantage\(^4\) and no refinements of subgame perfect equilibrium are needed to obtain the result.

2 The model

2.1 Weighted majority games

Consider a legislature in which \(n\) parties are represented. We denote these as \(N = \{1, 2, \ldots, n\}\). There is a budget of size 1 to be divided by majority rule. Each party \(i\) has \(\omega_i\) votes, and a quota of \(q\) is needed for a majority. The pair \([q; (\omega_i)_{i \in N}]\) is a weighted majority game. Notice that the game is not affected if weights and quota are multiplied by the same positive constant.

Given a vector \(x \in \mathbb{R}^N\) and a coalition \(S \subset N\), we denote as \(x_S\) the sum of the coordinates of the members of \(S\), \(x_S := \sum_{i \in S} x_i\).

A coalition \(S \subset N\) is winning if \(\omega_S \geq q\); it is minimal winning if it is winning and no \(T \subsetneq S\) is winning. We denote as \(\Omega(\omega)\) the set of all winning coalitions, and as \(\Omega^m(\omega)\) the set of all minimal winning coalitions.

A dummy player is a player who does not belong to any minimal winning coalition.

A weighted majority game is constant-sum if \(S \in \Omega(\omega) \iff N \setminus S \notin \Omega(\omega)\) for all \(S\). It admits an equivalent homogeneous representation if there exists a vector of votes \((\omega^1, \ldots, \omega^n)\) and a quota \(q^h\) such that \(\Omega^m(\omega) = \Omega^m(\omega^h) = \{S \subset N : \omega_S^h = q^h\}\). A weighted majority game that admits an

\(^4\)The empirical evidence lies somewhere in between. A large body of empirical studies (see e.g. Browne and Franklin [8], Schofield and Laver [26], Warwick and Druckman [31]) find little or no advantage to being formateur: a party’s share of cabinet posts is nearly proportional to its share of legislative seats in the governing coalition. Ansolabehere et al. [1] find a formateur advantage using voting weights instead of seat shares, though this advantage is well below the value predicted by the Baron-Ferejohn model. Interestingly, experiments on majority games do not find as sharp a behavioral difference between both types of procedures as the theory predicts (Fréchette et al. [14]).
equivalent homogeneous representation is called a *homogeneous game*.

Homogeneous representations are important because they better reflect the real bargaining power of the parties. For example, in the game \([5; 4, 3, 2]\), any two parties have a majority of votes, and the homogeneous representation \([2; 1, 1, 1]\) reflects this symmetry.

Homogeneous representations do not always exist and when they exist they may not be unique. For example, \([5; 3, 2, 2, 1]\) and \([7; 4, 3, 3, 1]\) are two homogeneous representations of the same game. Peleg [24] shows that constant-sum homogeneous games have a unique homogeneous representation (up to multiplication by a positive constant and to the weight that is assigned to dummies, which may be 0 or a sufficiently small number).

### 2.2 The bargaining procedure

Let there be a constant-sum homogeneous game, and \([q; (w_i)_{i \in N}]\) a homogeneous representation of the game. There is a budget of size 1 to divide. Party \(i\)'s utility function is \(u_i(x_i)\), where \(x_i\) is \(i\)'s share of the budget, and \(u'_i(x_i) > 0\) for all \(x_i\). Bargaining proceeds as follows. Parties move in decreasing order of weight. We label the parties in this order, so that party 1 moves first, followed by party 2, etc. \(^5\)

Each party \(i\) makes a demand \(d_i\), following the order of play, where \(d_i \in [0, 1]\) is the share of the budget party \(i\) claims. If, after party \(i\) makes its demand, there exists a winning coalition \(S \subset \{j : j \leq i\}\) such that \(d_S \leq 1\), party \(i\) has the additional choice of forming coalition \(S\), in which case payoffs are distributed according to the demands made. If there is more than one possible \(S\), party \(i\) decides which one is formed. If party \(n\) forms no coalition, the game ends with each party getting zero. \(^6\)

\(^5\)Of course, since it may be the case that \(\omega_i = \omega_j\) for some parties \(i\) and \(j\), the bargaining procedure is uniquely defined only up to a permutation of parties with the same weight. Notice that selection by decreasing seat share is a particular case of selection by decreasing voting weight.

\(^6\)Any finite number of bargaining rounds \(T\) would lead to the same results.
procedure is identical to that in [11], except that the order of moves must be by decreasing weight rather than randomly determined at the start.

Given \( i \in N \), we denote as \( P_i \) the set of predecessors of \( i \). Namely:

\[
P_i := \{ j \in N : j < i \}.
\]

As it will become clear from the analysis, dummy players must get 0 in equilibrium, so for simplicity we assume there are no dummy players. In what follows, we will take \( [q; (w_i)_{i \in N}] \) to be the homogeneous representation with \( \omega_n = 1 \); i.e. the weakest party has exactly 1 vote. Under these circumstances, every party in a constant-sum homogeneous game has a positive integer number of votes. Furthermore:

**Lemma 1** Let \( [q; (w_i)_{i \in N}] \) be a constant-sum homogeneous game. Then, \( \omega_N = 2q - 1 \).

**Proof.** Because \( n \) is not a dummy player, there exists \( S \in \Omega^m(\omega) \) such that \( n \in S \). Homogeneity implies \( \omega_S = q \). Because \( S \in \Omega^m(\omega) \), \( S \{n\} \) must be losing. Since the game is constant-sum, \( (N \setminus S) \cup \{n\} \in \Omega(\omega) \). Moreover, by deleting the weakest party (i.e. party \( n \)) we obtain a losing coalition \( N \setminus S \). Thus, \( (N \setminus S) \cup \{n\} \in \Omega^m(\omega) \). So, \( \omega_{(N \setminus S) \cup \{n\}} = q \) and \( \omega_{N \setminus S} = q - 1 \). Hence

\[
\omega_N = \omega_S + \omega_{N \setminus S} = q + q - 1 = 2q - 1.
\]

\[\square\]

**Corollary 1** Let \( [q; (w_i)_{i \in N}] \) be a constant-sum homogeneous game. Then, \( S \) is maximal losing (i.e. \( S \) is losing and no \( T \supseteq S \) is losing) iff \( \omega_S = q - 1 \).

**Proof.** Since \( (N, v) \) is constant-sum, \( S \) is maximal losing iff \( N \setminus S \in \Omega^m(\omega) \). Homogeneity then implies \( \omega_{N \setminus S} = q \) and thus, under Lemma 1,

\[
\omega_S = \omega_N - \omega_{N \setminus S} = 2q - 1 - q = q - 1.
\]

\[\square\]
Lemma 2 Let \([q; (w_i)_{i \in N}]\) be a weighted majority game. Then, there is a party \(i\) such that \(P_{i+1} \in \Omega^m(\omega)\).

Proof. Suppose this was not the case. Consider the smallest index \(i\) such that \(S = \{1, \ldots, i\}\) is a winning coalition. There is a minimal winning coalition \(S' \subset S\), and \(S'\) is obtained from \(S\) by deleting at least one party \(j < i\). However, this is impossible because by assumption \(\{1, \ldots, i-1\}\) is a losing coalition, and, since \(w_j \geq w_i\) for all \(j < i\), this coalition has at least as many votes as \(S'\). ■

Lemma 2 does not hold for arbitrary orders of the parties. For example, if we take the game \([3; 2, 1, 1, 1]\) and order the parties in such a way that the party with 2 votes is in the third place, no set of parties \(\{1, \ldots, i\}\) is a minimal winning coalition. If the parties play the game in this order, the party that moves first cannot get a positive payoff for any demand, and this leads to a continuum of subgame perfect equilibria.

Theorem 1 Let \([q; (w_i)_{i \in N}]\) be a constant-sum homogeneous game. Suppose parties play a demand commitment game in decreasing order of weight. Then in any subgame perfect equilibrium the minimal winning coalition of Lemma 2 forms with each party \(i\) demanding \(\frac{w_i}{q}\).

Proof. See Appendix. ■

The equilibrium strategies are roughly as follows (for a formal description see Appendix). Given the demands of the parties that have moved so far, party \(i\) determines two things: the optimal coalition to be (eventually) formed and the optimal demand to make.

In general, the optimal coalition \(S\) will control exactly \(q\) votes. This coalition will generally include some parties that have moved before \(i\), as well as some parties moving after \(i\). Since \(T = S \cap P_i\) is a group of parties that have already formulated a demand, \(1 - d_T\) is the benefit from buying the votes of the parties in \(T\); this benefit will be shared by the parties in \(S \setminus T\). Buying less votes leads to a higher benefit, but more votes from parties
moving after $i$ will be needed to complete a winning coalition. The coalition $S$ is chosen such that the average benefit per vote, $\frac{1-d_T}{q-\omega_T}$, is maximized.

The optimal demand for party $i$ will normally be $d_i = \omega_i \frac{1-d_T}{q-\omega_T}$, that is, party $i$ will claim a share of the benefit proportional to its number of votes. Only in some subgames outside the equilibrium path can party $i$ demand more than a proportional share.

Below we present a worked out example.

**Example 1** Suppose there are five parties, with $3, 2, 2, 1$ and $1$ votes respectively, and the quota is $5$. Any subgame perfect equilibrium of the demand commitment game is such that coalition $\{1, 2\}$ forms with $d_1 = \frac{3}{5}$ and $d_2 = \frac{2}{5}$.

**Proof.** At stage 5, party 5 faces a vector of demands $(d_1, d_2, d_3)$. It has three choices:

a) Form coalition $\{1, 4, 5\}$ and get $1 - d_1 - d_4$.

b) Form coalition $\{2, 3, 5\}$ and get $1 - d_2 - d_3$.

c) Form no coalition and get 0.\(^7\)

Suppose one of the first two options is optimal. Then party 5 will form coalition $\{1, 4, 5\}$ if $1 - d_1 - d_4 \geq 1 - d_2 - d_3$, or $d_4 \leq d_2 + d_3 - d_1$. Ties are broken in favor of forming the coalition that includes party 4, to guarantee that party 4 has a best response in the previous stage. Hence the maximum demand 4 can make and still get into a coalition with 5 is $d_4 = d_2 + d_3 - d_1$.

At stage 4, party 4 faces a vector of demands $(d_1, d_2, d_3)$. It can form coalition $\{2, 3, 4\}$ or make a demand that will lead to $\{1, 4, 5\}$. It forms $\{2, 3, 4\}$ if $1 - d_2 - d_3 \geq d_2 + d_3 - d_1$, or

$$1 - d_2 - d_3 \geq \frac{1 - d_1}{2}.$$

\(^7\)In order to simplify the proof, we only consider minimal winning coalitions. A coalition like $\{1, 2, 5\}$ could be relevant if $d_2 \leq d_4$ and $1 - d_1 - d_2 > 0$, but this requires player 2 to have acted irrationally: player 2 would have been better-off by setting $d_2 = 1 - d_1$ and forming coalition $\{1, 2\}$.
Thus, party 4 is effectively comparing the average benefit associated to buying the votes of 2 and 3 (in which case 1 vote is enough to complete a winning coalition) or the votes of 1 (in which case 2 votes are needed to complete a winning coalition and 4 must share the benefit with 5).

From the inequality above, the maximum demand party 3 can make at the previous stage and still induce \{2, 3, 4\} is

\[ d_3 = \frac{1 - 2d_2 + d_1}{2}. \]

At stage 3, party 3 faces a vector of demands \((d_1, d_2)\). It can form coalition \{1, 3\} or make a demand that will induce \{2, 3, 4\}. It makes a demand if \(\frac{1 - 2d_2 + d_1}{2} \geq 1 - d_1\) or

\[ \frac{1 - d_2}{3} \geq \frac{1 - d_1}{2}. \]

Again, party 3 may buy the votes of party 1 (in which case 2 votes are required to complete a winning coalition), or the votes of party 2 (in which case 3 votes are required to complete a winning coalition). It chooses the alternative with the highest average benefit.

The maximum demand party 2 can make in the previous stage and still induce coalition \{2, 3, 4\} is

\[ d_2 = \frac{3d_1 - 1}{2}. \]

At stage 2, party 2 compares \(1 - d_1\) and \(\frac{3d_1 - 1}{2}\). It forms \{1, 2\} if \(\frac{3d_1 - 1}{2} \leq 1 - d_1\), or \(d_1 \leq \frac{3}{5}\). This inequality can be rewritten as \(\frac{1 - d_1}{2} \geq \frac{1}{5}\) (where \(\frac{1}{5}\) is the average benefit of buying no votes).

Anticipating this, party 1 sets \(d_1 = \frac{2}{5}\). Party 2 will then set \(d_2 = \frac{2}{5}\) and coalition \{1, 2\} is formed.

3 Concluding remarks and discussion

We have presented a demand bargaining model that makes sharp predictions regarding coalition formation and payoff division. The model can be
extended to any finite horizon, and its predictions are independent of the
discount factors and the risk attitudes of the parties.

The proportional payoff prediction of our model is intuitive in the ab-
sence of policy preferences. Proportional payoffs are also predicted by many
solution concepts like von Neumann-Morgenstern’s [29] main simple solu-
tion, the set of balanced aspirations (Cross, [10]), the competitive solution
(McKelvey et al., [17]) and the demand bargaining set (Morelli and Mon-
tero, [21]). Those cooperative solution concepts also require the game to be
homogeneous and constant-sum in order for payoffs to be proportional.  

The coalition that forms is the minimal winning coalition with the small-
est number of parties. One may ask whether proportional payoffs can be
achieved for an arbitrary minimal winning coalition by choosing the order
of moves appropriately. The answer is negative: for the game \([5; 3, 2, 2, 1, 1]\),
there is no order of moves for which coalition \(\{1, 4, 5\}\) forms with a propor-
tional payoff division. There are three types of possible orders for which the
parties in this coalition move first: \([31122]\), \([13122]\) and \([11322]\). It can be
shown that the first mover gets the whole budget in order \([31122]\), whereas
in the other two orders the first mover gets half of the budget.

If the game is not constant-sum and homogeneous, proportionality may
break down. In some cases, this is due to the presence of a party that can
be ”held hostage” by others, as pointed by Morelli [20].

Example 2 There are four parties, with 3, 2, 2 and 1 votes respectively.
The quota is 5. If the parties play a demand commitment game in decreasing
order, any subgame perfect equilibrium results in coalition \(\{1, 2\}\) with \(d_1 = \frac{1}{2}\)
and \(d_2 = \frac{1}{2}\).

Party 4 is helpless because there is only one minimal winning coalition it

\(^8\)Proportionality results can also be obtained in the context of the Baron-Ferejohn
model (see Montero, [18]). However, this proportionality is \textit{ex ante} (ex post the proposer
obtains more than half of the total payoff) and in order to hold generally it requires the
recognition probabilities to be themselves proportional.
can form. Knowing this, party 3 will either form a coalition with 1 and get $1 - d_1$, or set $d_3 = 1 - d_2$. Party 2 can then form a coalition with 1 (obtaining $1 - d_1$) or set $d_2 = d_1$ and induce coalition $\{2, 3, 4\}$. Anticipating this, party 1 sets $d_1 = \frac{1}{2}$. The game $[5; 3, 2, 2, 1]$ has many homogeneous representations, but in none of them do parties 1 and 2 have the same number of votes.

Proportionality can break down even if no party can be held hostage by others, as the following example illustrates.

**Example 3** Consider the game $[7; 4, 3, 2, 2, 1, 1]$. If the parties play a demand commitment game in decreasing order, any subgame perfect equilibrium results in coalition $\{1, 2\}$ with $d_1 = d_2 = \frac{1}{2}$.

The game above is constant-sum but not homogeneous. None of the parties can be held hostage by others: given any two parties, each of them can form a minimal winning coalition that does not include the other. Moreover, coalition $\{1, 2\}$ has exactly 7 votes. Nevertheless, proportionality fails because $\{1, 3, 4\}$ and $\{2, 3, 4\}$ are both minimal winning coalitions. From the point of view of parties 3 and 4, parties 1 and 2 are equally valuable even though they have a different number of votes. If the turn reaches party 3, which of the two coalitions forms will depend on whether $d_1$ is higher or lower than $d_2$. Anticipating this, party 2 has two options: it can form a coalition with 1 and get $1 - d_1$, or set $d_2 = d_1$ and induce coalition $\{2, 3, 4\}$. Party 2 will form a coalition if $1 - d_1 \geq d_1$, or $d_1 \leq \frac{1}{2}$.

### 4 Appendix: Proof of Theorem 1

The result trivially follows if there is a veto player. In constant-sum games, a veto player must be a dictator, thus $\omega_1 = q$, and $d_1 = 1$ would be the equilibrium outcome. We will assume from now on that $\omega_i < q$ for all $i$.

We denote as $\mathbb{B}(d, i)$ with $i \in N$ and $d \in \mathbb{R}^N$ the subgame which begins when it is party $i$’s turn, facing a vector $d$ of demands. At subgame $\mathbb{B}(d, i)$, party $i$ will determine the optimal winning coalition $S \ni i$ to be formed,
and will formulate a demand $d_i$ that will lead to $S$ being formed. We will show how party $i$ determines which coalition is optimal as well as how the optimal coalition can be induced by the choice of $d_i$.

Suppose we are in $B(d, i)$, and party $i$ plans to make a demand in the belief that a coalition $S \in \Omega(\omega)$ with $i \in S$ will be formed. This coalition should include some parties from $N \setminus P_i$ (party $i$ and possibly parties that move after it) and may also include some predecessors from $P_i$. Let $\alpha$ be the number of votes controlled by parties in $S \cap (N \setminus P_i)$. Then, the parties in $S \cap P_i$ should control at least $q - \alpha$ votes. We denote as $b(i, \alpha)$ the maximum benefit that can be achieved by buying these $q - \alpha$ votes from parties in $P_i$.

$$b(i, \alpha) := \max \{1 - d_T : T \subset P_i, \omega_T \geq q - \alpha\}.$$  

Party $i$ can calculate $b(i, \alpha)$ for every feasible value of $\alpha$. Notice that not all integers between 0 and $q$ are feasible for every player. First, $\alpha$ cannot be so small that even the votes of all the parties in $P_i$ would not suffice. Let

$$\gamma^i_0 := q - \omega_P.$$  

In order for $b(i, \alpha)$ to exist we need $\alpha \geq \gamma^i_0$.

Since party $i$ must be in $S$, it seems reasonable to require $\alpha \geq \omega_i$ as well. The next lemma shows that this is unnecessary: there is no positive benefit from buying more than $q - \omega_i$ votes.

**Lemma 3** Let $\gamma^i_0 \leq \alpha < \omega_i$ and suppose no party $j < i$ has made a strictly dominated choice of $d_j$. Then, $b(i, \alpha) \leq 0$. Moreover, $b(i, \alpha) = 0$ implies $b(i, \omega_i) \geq 0$.

**Proof.** Let $T \subset P_i$ such that $\omega_T \geq q - \alpha$. Since $\alpha < \omega_i$, we have

$$\omega_{T \cup \{i\}} = \omega_T + \omega_i > \omega_T + \alpha \geq q.$$  

Hence, since the game is homogeneous, $T \cup \{i\}$ cannot be a minimal winning coalition. Moreover, party $i$ is the party with the least votes in
$T \cup \{i\}$, thus coalition $T$ should be winning. This means that either $d_T \geq 1$ (implying $b(i, \alpha) \leq 0$) or $d_T < 1$, in which case the smallest party in $T$ (party $j$) would have been strictly better-off by setting a higher demand and forming a coalition, regardless of the actions of the parties moving after $j$.

Moreover, when $b(i, \alpha) = 0$, $b(i, \omega_i) \geq 0$ follows from the fact that $b(i, \cdot)$ is nondecreasing in the second variable. ■

We will eliminate strictly dominated strategies, thus in all the subgames we study it will be the case that $b(i, \alpha) \leq 0$ for $\gamma_0^i \leq \alpha < \omega_i$. Otherwise the turn would never have reached party $i$.

Since there is no positive benefit from buying more than $q - \omega_i$ votes, and (given that there is no benefit left to be divided) the particular value of $\alpha$ is irrelevant if $b(i, \alpha) = 0$, any lower bound between 0 and $\omega_i$ can be equivalently used by party $i$. We take $\alpha$ to be greater or equal to:

$$\gamma_i^i := \max \{1, \gamma_0^i\}.$$ 

Moreover, party $i$ is constrained by the number of votes owned by parties in $N \setminus P_i$. Thus, $\alpha$ must be smaller or equal to

$$\delta_i^i := \omega_{N \setminus P_i}.$$ 

Notice that $\delta_{i+1}^i = \delta_i^i - \omega_i$ for all $i < n$. Also, $\omega_i < q$ implies $\gamma_0^i \leq \delta_{i+1}^i$. It follows from lemma 4.9 in Ostmann [23] that $\omega_i \leq \delta_{i+1}^i$ for all $i < n$, thus $\gamma_i^i \leq \delta_{i+1}^i$ for all $i < n$.

For party 1 only $\alpha \geq q$ is feasible and $b(1, \alpha) = 1$ for all $\alpha \geq q$. For party $n$, only $\alpha = 1$ is feasible and $b(n, 1)$ is simply $n$'s payoff from buying the votes of one of the cheapest coalitions controlling at least $q - 1$ votes.

The following lemma shows how $b(i + 1, \alpha)$ is determined from $b(i, \cdot)$ and $d_i$. It may be the case that, having $\alpha$ votes in its pocket, party $i + 1$ cannot

\[9\text{A lower bound of 1 has the advantage of being independent of } i \text{ and allowing division by all values of } \alpha, \text{ but the proof can be adapted to any other choice.}\]
form a winning coalition without party $i$. Then $b(i + 1, \alpha) = b(i, \alpha + \omega_i) - d_i$ irrespective of $d_i$. Otherwise party $i + 1$ will compare the best coalition that includes $i$ with the best coalition that does not include $i$. Given that $i$ is included in the coalition, $i + 1$ needs to buy the remaining votes $(q - (\alpha + \omega_i))$ from $P_i$, and the best way to do this leads to a benefit of $b(i, \alpha + \omega_i)$; after paying $d_i$, there would be $b(i, \alpha + \omega_i) - d_i$ left. Without party $i$, the maximum benefit from buying $q - \alpha$ votes without buying $i$’s votes is precisely $b(i, \alpha)$. Party $i$ will then be included if $d_i$ is sufficiently low.

Whether $d_i$ is sufficiently low depends on the demands of the parties in $P_i$. Because parties may be complements, in some cases no positive demand by $i$ would be low enough, as the following example illustrates.

Consider the game $[10; 7, 3, 3, 3, 1, 1, 1]$. Let $i = 3$, $i + 1 = 4$. We have $b(3, 7) = \max(1 - d_1, 1 - d_2)$ and $b(3, 4) = 1 - d_1$. Having 7 votes in its pocket, party 3 may buy the votes of either party 1 (with a benefit of $1 - d_1$) or party 2 (with a benefit of $1 - d_2$). On the other hand, having only 4 votes, party 3 must buy the votes of party 1, with a benefit of $1 - d_1$.

If party 4 wants to compute $b(4, 4)$ it compares $1 - d_1$ and $1 - d_2 - d_3$. Thus in this particular case parties 2 and 3 are complements. If $d_3$ is high, then $b(4, 4) = 1 - d_1$, which is precisely $b(3, 4)$. If both $d_3$ and $d_2$ are sufficiently low, then $b(4, 4) = 1 - d_2 - d_3$ and $b(3, 7) = 1 - d_2$, hence $b(4, 4) = b(3, 4 + \omega_3) - d_3$. If $d_2 > d_1$, no positive value of $d_3$ is sufficiently low.

**Lemma 4** Suppose we are in $\mathbb{B}(d, i + 1)$. Let $\alpha$ such that $\gamma_{i+1}^0 \leq \alpha \leq \delta_{i+1}$. Then $\gamma_{i}^0 \leq \alpha + \omega_i \leq \delta_i$ and furthermore:

- a) if $\alpha < \gamma_{i}^0$, then $b(i + 1, \alpha) = b(i, \alpha + \omega_i) - d_i$;
- b) if $\alpha \geq \gamma_{i}^0$, then $b(i, \alpha)$ exists and

$$b(i + 1, \alpha) = \max \{b(i, \alpha), b(i, \alpha + \omega_i) - d_i\}.$$  

**Proof.** We have to prove that $\gamma_{i}^0 \leq \alpha + \omega_i \leq \delta_i$. It is straightforward:

- $\alpha \leq \delta_{i+1} \Rightarrow \alpha + \omega_i \leq \delta_{i+1} + \omega_i = \delta_i$.
- $\alpha \geq \gamma_{i+1}^0 = q - \omega_{P_{i+1}} \Rightarrow \alpha + \omega_i \geq q - \omega_{P_{i+1}} + \omega_i = q - \omega_{P_i} = \gamma_{i}^0$.  

14
a) If $\alpha < \gamma_i^0$, every $T \subset P_{i+1}$ with $\omega_T \geq q - \alpha$ satisfies $i \in T$. Then:

$$b(i + 1, \alpha) = \max_{T \subset P_{i+1}: \omega_T \geq q - \alpha} (1 - d_T) = \max_{T \subset P_{i+1}: \exists i \in T, \omega_T \geq q - \alpha} (1 - d_T)$$

$$= \max_{T \subset P_i: \omega_T \geq q - \omega_i} (1 - d_T) = (1 - d_T) - d_i = b(i, \alpha + \omega_i) - d_i.$$

b) If $\alpha \geq \gamma_i^0$, $b(i, \alpha)$ is well defined and

$$b(i + 1, \alpha) = \max_{T \subset P_{i+1}: \omega_T \geq q - \alpha} (1 - d_T)$$

$$= \max \left\{ \max_{T \subset P_{i+1}: \exists i \in T, \omega_T \geq q - \alpha} (1 - d_T), \max_{T \subset P_{i+1}: \exists i \in T, \omega_T \geq q - \alpha} (1 - d_T) \right\}$$

$$= \max \left\{ \max_{T \subset P_i: \omega_T \geq q - \omega_i} (1 - d_T), \max_{T \subset P_i: \omega_T \geq q - \omega_i - \omega_i} (1 - d_T) - d_i \right\}$$

$$= \max \left\{ b(i, \alpha), b(i, \alpha + \omega_i) - d_i \right\}.$$

We have defined the best way to form a coalition that contains $\alpha$ votes from $N \setminus P_i$ and at least $q - \alpha$ votes from $P_i$. It remains to choose the optimal value of $\alpha$, and the optimal demand $d_i$.

We denote as $\Sigma^i$ the set of values between $\gamma_i$ and $\delta_i$ that maximize $b(i, \alpha)/\alpha$. Thus:

$$\Sigma^i := \arg \max_{\gamma_i \leq \alpha \leq \delta_i} \frac{b(i, \alpha)}{\alpha}$$

The next lemma shows that the only interesting bargaining occurs when $b(i, \sigma^i) \geq 0$ for some/all $\sigma^i \in \Sigma^i$.

**Lemma 5** Suppose we are in a subgame perfect equilibrium (SPE) of $B(d, i)$. If $b(i, \sigma^i) < 0$ for some/all $\sigma^i \in \Sigma^i$, then every party gets zero.

**Proof.** Since $b(i, \sigma^i)/\sigma^i$ is maximum, we deduce that $b(i, \alpha) < 0$ for every $\alpha \geq \gamma_i$. The same occurs for $\alpha = 0$ since $b(i, \alpha)$ is nondecreasing in $\alpha$. This means that no winning coalition can be formed. $\blacksquare$

---

$^{10}$Of course, $b(i, \sigma^i) \geq 0$ for some $\sigma^i \in \Sigma^i$ implies $b(i, \sigma^i) \geq 0$ for all $\sigma^i \in \Sigma^i$. 

15
Thus, if \( b(i, \sigma^i) < 0 \) for some/all \( \sigma^i \in \Sigma^i \), party \( i \) formulates an arbitrary demand and the game eventually ends with no coalition being formed.

From now on, we will assume that \( b(i, \sigma^i) \geq 0 \) for all \( \sigma^i \in \Sigma^i \). We will show that in equilibrium party \( i \) always chooses some \( \alpha \in \Sigma^i \).

The following lemma shows that all values of \( \alpha \) between \( \delta^i + 1 \) and \( \delta^i + \omega_i = \delta^i \) lead to the same \( b(i, \alpha) \). The extra votes are not valuable because they are not enough to replace any party from \( P_i \).

For example, in the game \([10; 7, 3, 3, 3, 3, 3, 1, 1, 1] \), \( \delta^4 = 6 \) and \( \delta^5 = 3 \). Consider the situation of party 4. If it takes \( \alpha = 4 \), there are two ways to form a winning coalition: buying the votes of party 1, or buying the votes of parties 2 and 3. Thus, \( b(4, 4) = \max(1 - d_1, 1 - d_2 - d_3) \). If instead it takes \( \alpha = 5 \) or \( \alpha = 6 \), exactly the same parties are needed: none of party 4’s predecessors can be dispensed with despite the extra votes.

**Lemma 6** Suppose we are in the subgame \( B(d, i) \). Then

\[
\{ T \subset P_i : \omega_T \geq q - (\delta^i + 1 + \alpha) \} = \{ T \subset P_i : \omega_T \geq q - \delta^i \}
\]

for all \( \alpha = 1, 2, \ldots, \omega_i \).

**Proof.** “\( \subset \)” Let \( T \subset P_i \) such that \( \omega_T \geq q - (\omega_{N \setminus P_i+1} + \omega_i + \alpha) \). Then

\[
\omega_T \geq q - (\omega_{N \setminus P_i} - \omega_i + \alpha) = q - \omega_{N \setminus P_i} + (\omega_i - \alpha) \geq q - \omega_{N \setminus P_i}.
\]

“\( \supset \)” Let \( T \subset P_i \) such that \( \omega_T \geq q - \omega_{N \setminus P_i} \). Then, \( T \cup (N \setminus P_i) \) is winning and contains party \( i \). There are two possibilities:

- **T \cup (N \setminus P_i) \setminus \{i\} = T \cup (N \setminus P_{i+1})** is also winning. Then, \( \omega_T \geq q - \omega_{N \setminus P_{i+1}} \) and the result is proved.

- **T \cup (N \setminus P_{i+1})** is losing. Then, since the game is constant-sum, we conclude that its complement, \((N \setminus T) \cap P_{i+1}\), is winning and contains party \( i \) as the weakest member. By taking out party \( i \), we obtain the coalition \((N \setminus T) \cap P_i\) which is losing (since its complement \( T \cup (N \setminus P_i)\)
is winning). Thus, \((N \setminus T) \cap P_{i+1}\) is minimal winning and \(T \cup (N \setminus P_{i+1})\) is maximal losing. Hence, under Corollary 1:

\[
\omega_T = q - \omega_{N \setminus P_{i+1}} - 1 \geq q - \omega_{N \setminus P_{i+1}} - \alpha.
\]

\[\blacksquare\]

**Corollary 2** In \(B(d, i)\), we have \(b(i, \delta^{i+1} + \alpha) = b(i, \delta^i)\) for all \(\alpha = 1, 2, \ldots, \omega_i\).

Moreover, for all \(\sigma^i \in \Sigma^i\), if \(b(i, \sigma^i) > 0\),

\[
\sigma^i > \delta^{i+1} \implies \sigma^i = \delta^{i+1} + 1.
\]

**Proof.** Under Lemma 6, it is clear that \(b(i, \delta^{i+1} + \alpha) = b(i, \delta^i)\) for all \(\alpha = 1, 2, \ldots, \omega_i\), since they minimize \(d_T\) on the same coalitions \(T\). Hence, if \(b(i, \delta^i) > 0\),

\[
\frac{b(i, \delta^{i+1} + \alpha)}{\delta^{i+1} + \alpha} < \frac{b(i, \delta^{i+1} + 1)}{\delta^{i+1} + 1}
\]

for all \(\alpha = 2, \ldots, \omega_i\) and thus the maximum is \(b(i, \delta^{i+1} + 1) / (\delta^{i+1} + 1)\). Let \(\sigma^i \in \Sigma^i\) such that \(\sigma^i > \delta^{i+1}\). Since \(\sigma^i = \delta^{i+1} + \alpha\) for some \(\alpha = 1, 2, \ldots, \omega_i\), we conclude the result. \[\blacksquare\]

Now we define what will turn out to be the maximum demand party \(i\) can make at \(B(d, i)\). This depends on what party \(i + 1\) can achieve without party \(i\). If party \(i + 1\) decides to exclude party \(i\), it is in a similar situation to party \(i\) except that it has less feasible values for \(\alpha\). It will be choosing an \(\alpha\) between \(\gamma^i\) and \(\delta^{i+1}\), and the maximum benefit from buying \(q - \alpha\) votes without party \(i\) is precisely \(b(i, \alpha)\). We define \(T^i\) as the set of values between \(\gamma^i\) and \(\delta^{i+1}\) that maximize \(b(i, \alpha) / \alpha\) (recall that \(\gamma^i \leq \delta^{i+1}\), so the interval is nonempty).

\[
T^i := \arg \max_{\gamma^i \leq \alpha \leq \delta^{i+1}} \frac{b(i, \alpha)}{\alpha}.
\]

Let \(\tau^i \in T^i\). Because \(\gamma^i \leq \tau^i\) and \(\tau^i \leq \delta^{i+1} < \delta^i\), \(\frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i, \sigma^i)}{\sigma^i}\) for all \(\sigma^i \in \Sigma^i\).
For any values of $\sigma^i \in \Sigma^i$ and $\tau^i \in T^i$, we define

$$d_i^* := \begin{cases} 
\frac{\omega_i b(i, \sigma^i)}{\sigma^i} & \text{if } \sigma^i \leq \delta^{i+1} \\
b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\sigma^i} b(i, \tau^i) & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) \geq 0 \\
b(i, \sigma^i) & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) < 0.
\end{cases} \quad (1)$$

Intuitively, party $i$ wants to form a coalition with $\alpha = \sigma^i$ and to claim as much of $b(i, \sigma^i)$ as possible, but it must take into account that party $i+1$ can obtain a benefit per vote of $\frac{b(i, \tau^i)}{\tau^i}$ without party $i$. If $b(i, \tau^i) < 0$, party $i+1$ is helpless and party $i$ can claim the whole $b(i, \sigma^i)$. Otherwise, party $i$ must allow a benefit of $\frac{b(i, \tau^i)}{\sigma^i}$ per vote for the parties moving after it that will provide the remaining $\sigma^i - \omega_i$ votes, thus $d_i = b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\sigma^i} b(i, \tau^i)$. Furthermore, if $\sigma^i \leq \delta^{i+1}$, party $i+1$ can obtain a benefit of $\frac{b(i, \sigma^i)}{\sigma^i}$ per vote without resorting to buying the votes of party $i$ and thus $\frac{b(i, \tau^i)}{\sigma^i} = \frac{b(i, \sigma^i)}{\sigma^i}$ and $b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\sigma^i} b(i, \tau^i) = \omega_i b(i, \sigma^i)$.

It is easy to prove that $d_i^*$ is independent of the particular choice of $\sigma^i \in \Sigma^i$ and $\tau^i \in T^i$. By definition, $\frac{b(i, \tau^i)}{\tau^i}$ and $\frac{b(i, \sigma^i)}{\sigma^i}$ are independent of the $\tau^i$ and $\sigma^i$ chosen. Also, $b(i, \tau^i) \geq 0$ for some $\tau^i \in T^i$ if and only if $b(i, \tau^i) \geq 0$ for all $\tau^i \in T^i$. If $b(i, \sigma^i) = 0$ for some $\sigma^i$, then $b(i, \sigma^i) = 0$ for all $\sigma^i$ and $b(i, \tau^i) \leq 0$ for all $\tau^i \in T^i$. Thus, $d_i^* = 0$ regardless of the choice of $\sigma^i$ and $\tau^i$. If $b(i, \sigma^i) > 0$, $d_i^*$ is the same for all $\sigma^i \leq \delta^{i+1}$.

If $\sigma^i > \delta^{i+1}$, $\sigma^i = \delta^{i+1} + 1$. If $\Sigma^i$ contains some $\sigma^i \leq \delta^{i+1}$ as well as $\sigma^i = \delta^{i+1} + 1$, $d_i^*$ will still be independent of the choice of $\sigma^i$ because in this case $T^i = \Sigma^i \backslash \{\delta^{i+1} + 1\}$, thus $\frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \sigma^i)}{\sigma^i}$.

We now present three lemmas that will be useful in proving that $d_i^*$ is the equilibrium demand of party $i$. As we will show in proposition 1, the set $\Sigma^i$ is used by $i$ to determine which coalition eventually forms. Each element of $\Sigma^i$ is associated to at least one coalition $T$. This coalition is the union of a (possibly empty) $S \subset P_i$ whose votes are to be bought ($S$ is one of the cheapest coalitions that control at least $q - \sigma^i$ votes) and a set of consecutive players ($i$ and possibly some successors) controlling $\sigma^i$ votes. The main

\[\text{If } b(i, \sigma^i) > 0, S \text{ must be losing, otherwise some party has made a strictly dominated}\]
The result of the lemmas is that, if \( \sigma^i > \omega_i \) for some \( \sigma^i \in \Sigma^i \) (so that party \( i \) cannot form \( T \) but must induce other parties to form it by an appropriate choice of \( d_i \)), \( d_i^* \) is the highest demand that leads to \( T \) being also associated to an element of \( \Sigma^{i+1} \).

The first step will be to show that party \( i \) by formulating a sufficiently low demand \( (d_i \leq d_i^*) \) can guarantee that \( \sigma^i - \omega_i \) is one of the values of \( \alpha \) that maximize the benefit per vote for party \( i + 1 \), and that at least one of the ways of buying \( q - (\sigma^i - \omega_i) \) votes involves buying the votes of party \( i \). Lemma 7 takes us close to this result. This lemma establishes that \( \sigma^i - \omega_i \) is a feasible value of \( \alpha \) for party \( i + 1 \), that one of the best ways to buy \( \sigma^i - \omega_i \) votes from \( P_{i+1} \) involves buying the votes of party \( i \) \( b(i+1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i \) and some inequalities that will help in proving that \( \sigma^i - \omega_i \in \Sigma^{i+1} \). Essentially, the inequalities say that buying exactly \( q - (\sigma^i - \omega_i) \) votes leads to a benefit per vote at least as high as the best option that does not involve buying the votes of party \( i \); the comparison with other options that involve party \( i \) will be made in lemma 8.

Notice that \( \sigma^i > \omega_i \) for some \( \sigma^i \in \Sigma^i \) implies \( i < n \), because \( \omega_n = 1 \) and \( \gamma^n = \delta^n = 1 \).

**Lemma 7** Suppose we are in \( \mathbb{B}(d, i) \). If \( \sigma^i > \omega_i \) for some \( \sigma^i \in \Sigma^i \) and party \( i \) demands \( d_i \leq d_i^* \), then

\[
\gamma^{i+1} \leq \sigma^i - \omega_i \leq \delta^{i+1}
\]

(2)

\[
b(i+1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i
\]

(3)

and, given any \( \tau^i \in T^i \),

\[
\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} \geq \begin{cases} 
\frac{b(i, \sigma^i)}{\omega^i} & \text{if } \sigma^i \leq \delta^{i+1} \\
\frac{b(i, \tau^i)}{\omega^i} & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) \geq 0 \\
0 & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) < 0.
\end{cases}
\]

(4)

Furthermore, the inequality in (4) is strict iff \( d_i < d_i^* \).
Proof. Let $\sigma^i \in \Sigma^i$ such that $\sigma^i > \omega_i$. We first prove (2):

\[
\sigma^i \leq \delta^i \implies \sigma^i - \omega_i \leq \delta^i - \omega_i = \delta^{i+1}.
\]

\[
\sigma^i \geq \gamma^i \geq \gamma^i_0 = q - \omega_{P_i} \implies \sigma^i - \omega_i \geq q - \omega_{P_i} - \omega_i = q - \omega_{P_{i+1}} = \gamma^{i+1}.
\]

\[
\sigma^i > \omega_i \implies \sigma^i - \omega_i > 0 \implies \sigma^i - \omega_i \geq 1.
\]

Thus, $\sigma^i - \omega_i$ is a feasible value of $\alpha$ for party $i + 1$. Notice that for $b(i, \sigma^i) > 0$, homogeneity implies $\sigma^i - \omega_i \geq \omega_{i+1}$.

We prove now (3) and (4). Under Lemma 4a), (3) is true when $\sigma^i - \omega_i < \gamma^i_0$. Then (4) follows immediately by replacing $\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}$ by $\frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i}$ and then using $d_i \leq d^*_i$.\footnote{Actually, $\sigma^i - \omega_i < \gamma^i_0$ implies $\sigma^i \leq \delta^{i+1}$, so two of the three cases are void.} Assume then $\sigma^i - \omega_i \geq \gamma^i_0$. We have two cases:

1. If $\sigma^i \leq \delta^{i+1}$, then $d^*_i = \frac{\omega_i b(i, \sigma^i)}{\sigma^i - \omega_i}$. Since $\sigma^i \in \Sigma^i$, re-arranging terms,

\[
\frac{b(i, \sigma^i - \omega_i)}{\sigma^i - \omega_i} \leq \frac{b(i, \sigma^i)}{\sigma^i} \implies b(i, \sigma^i - \omega_i) \leq b(i, \sigma^i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}
\]

\[
\implies b(i, \sigma^i - \omega_i) \leq b(i, \sigma^i) - d_i.
\]

Hence, (3) follows under lemma 4b). Moreover

\[
\frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} = \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq \frac{b(i, \sigma^i) - \omega_i b(i, \sigma^i)}{\sigma^i - \omega_i}
\]

\[
= \frac{(\sigma^i - \omega_i) b(i, \sigma^i)}{\sigma^i (\sigma^i - \omega_i)} = \frac{b(i, \sigma^i)}{\sigma^i}
\]

with strict inequality iff $d_i < d^*_i$.

2. If $\sigma^i > \delta^{i+1}$, recall that $\gamma^i \leq \sigma^i - \omega_i \leq \delta^{i+1}$. Then for any $\tau^i \in T_i$

\[
\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
\]

Re-arranging terms,

\[
\frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \geq b(i, \sigma^i - \omega_i)
\]

\[
\implies b(i, \sigma^i - \omega_i) + b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \leq b(i, \sigma^i)
\]

\[
\implies b(i, \sigma^i - \omega_i) + d_i \leq b(i, \sigma^i).
\]
Hence, (3) follows under lemma 4b).

To show (4), we distinguish two subcases:

(a) If \( b(i, \tau^i) \geq 0 \), then
\[
\frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} = \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq \frac{b(i, \sigma^i) - b(i, \tau^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^i - \omega_i}
\]

\[
= \frac{b(i, \tau^i)}{\tau^i}
\]

with strict inequality iff \( d_i < d^*_i \).

(b) If \( b(i, \tau^i) < 0 \), then \( d^*_i = b(i, \sigma^i) \) and thus
\[
\frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} = \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq 0
\]

with strict inequality iff \( d_i < d^*_i \).

We now show that, if \( d_i < d^*_i \), all optimal (in the sense of maximizing benefit per vote) coalitions for party \( i + 1 \) involve party \( i \). If \( d_i = d^*_i \) there may be optimal coalitions for party \( i + 1 \) that do not involve \( i \); nevertheless, if we limit ourselves to coalitions involving party \( i \), exactly the same coalitions are optimal for party \( i \) and party \( i + 1 \).

**Lemma 8** Suppose we are in \( B(d, i + 1) \) and \( \sigma^i > \omega_i \) for some \( \sigma^i \in \Sigma_i \).

a) If \( d_i < d^*_i \), then for all \( \sigma^{i+1} \in \Sigma^{i+1} \) and all \( S \in \arg\max_{T \subset P_{i+1} \colon \omega_T \geq q - \sigma^{i+1}} (1 - d_T) \) it holds that \( i \in S \).

b) If \( d_i = d^*_i \), then \( \sigma^i - \omega_i \in \Sigma^{i+1} \). Moreover, \( S \in \arg\max_{T \subset P_i \colon \omega_T \geq q - \sigma^i} (1 - d_T) \) implies \( S \cup \{i\} \in \arg\max_{T \subset P_{i+1} \colon \omega_T \geq q - (\sigma^i - \omega_i)} (1 - d_T) \).

c) If \( d_i = d^*_i \), given \( \sigma^{i+1} \in \Sigma^{i+1} \) and \( S \in \arg\max_{T \subset P_{i+1} \colon \omega_T \geq q - \sigma^{i+1}} (1 - d_T) \), \( i \in S \) implies \( S \cap P_i \in \arg\max_{T \subset P_i \colon \omega_T \geq q - \sigma^i} (1 - d_T) \) for some \( \sigma^i \in \Sigma_i \).
Proof. a) Let $\sigma^{i+1} \in \Sigma^{i+1}$. Suppose there exists $S \in \arg \max_{T \subseteq P_{t+1}, \omega_T \geq q-\sigma^{i+1}} (1 - d_T)$ such that $i \notin S$. Then, $b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1})$. We see two cases:

1. If $\sigma^i \leq \delta^{i+1}$,
   \[
   \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1})}{\sigma^{i+1}} \leq \frac{b(i, \sigma^i)}{\sigma^i} \quad \text{(Lemma 7)} \quad < \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i},
   \]
   which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$.

2. If $\sigma^i > \delta^{i+1}$,
   \[
   \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1})}{\sigma^{i+1}} \leq \frac{b(i, \tau^i)}{\tau^i} \quad \text{(Lemma 7)} \quad < \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i},
   \]
   which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$.

b) Let $\alpha$ such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 4, either $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ or $b(i+1, \alpha) = b(i, \alpha)$. We have to prove that
   \[
   \frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
   \]
   If $b(i+1, \alpha) = b(i, \alpha)$, we proceed like in case a).
   If $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$, we have three cases:

1. If $\sigma^i \leq \delta^{i+1}$, then
   \[
   \frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - \omega_i b(i, \sigma^i)}{\sigma^i} \leq \frac{b(i, \sigma^i)(\alpha + \omega_i) - \omega_i b(i, \sigma^i)}{\sigma^i} = \frac{b(i, \sigma^i)}{\sigma^i} \quad \text{(Lemma 7)} \quad \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
   \]
2. If \( \sigma^i > \delta^{i+1} \) and \( b(i, \tau^i) \geq 0 \) for some/all \( \tau^i \in T^i \), then either \( \frac{b(i, \alpha + \omega_i)}{\alpha + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i} \) (if \( \alpha + \omega_i \leq \delta^{i+1} \)) or \( b(i, \alpha + \omega_i) = b(i, \sigma^i) \) (if \( \alpha + \omega_i > \delta^{i+1} \), by Corollary 2).

If \( \alpha + \omega_i \leq \delta^{i+1} \),
\[
\frac{b(i + 1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\
\leq \frac{\alpha + \omega_i}{\tau^i} b(i, \tau^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \\
= \frac{b(i, \tau^i)}{\tau^i} (\text{Lemma 7}) \frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
\]

If \( \alpha + \omega_i > \delta^{i+1} \),
\[
\frac{b(i + 1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\
= \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i).
\]

If \( b(i, \sigma^i) > 0 \), corollary 2 implies \( \sigma^i = \delta^{i+1} + 1 \). Then \( \alpha + \omega_i > \delta^{i+1} \) implies \( \alpha + \omega_i \geq \delta^{i+1} + 1 = \sigma^i \), or \( (\sigma^i - \omega_i) / \alpha \leq 1 \). If \( b(i, \sigma^i) = 0 \), \( b(i, \tau^i) = 0 \), implying \( \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) = \frac{b(i, \tau^i)}{\tau^i} \). In either case,
\[
\frac{b(i + 1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} (\text{Lemma 7}) \frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
\]

3. If \( \sigma^i > \delta^{i+1} \) and \( b(i, \tau^i) < 0 \) for some/all \( \tau^i \in T^i \), then either \( b(i, \alpha + \omega_i) < 0 \) (if \( \alpha + \omega_i \leq \delta^{i+1} \)) or \( b(i, \alpha + \omega_i) = b(i, \sigma^i) \) (if \( \alpha + \omega_i > \delta^{i+1} \), by Corollary 2).

If \( b(i, \alpha + \omega_i) < 0 \),
\[
\frac{b(i + 1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} < -\frac{d_i}{\alpha} \leq 0 \leq \frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
\]

If \( b(i, \alpha + \omega_i) = b(i, \sigma^i) \),
\[
\frac{b(i + 1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = 0 \leq \frac{b(i + 1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.
\]
We now prove the second statement. Let $S \in \arg\max_{T \subset P_i \omega_T \geq q - \sigma^i} (1 - d_T)$. We have to prove $b(i + 1, \sigma^i - \omega_i) = 1 - d_{S \cup (i)}$. Using (3),

$$b(i + 1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i = 1 - d_S - d_i = 1 - d_{S \cup (i)}.$$  

\[ \text{(c) Since } i \in S, b(i + 1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i, \text{ or} \]

\[ b(i, \sigma^{i+1} + \omega_i) = b(i + 1, \sigma^{i+1}) + d_i. \tag{5} \]

Let $\sigma^i > \omega_i$. We have shown that $\sigma^i - \omega_i \in \Sigma^{i+1}$, thus

$$b(i + 1, \sigma^{i+1}) \frac{\sigma^{i+1}}{\sigma^i} = b(i + 1, \sigma^i - \omega_i) \frac{\sigma^i}{\sigma^i - \omega_i}. \tag{6}$$

1. If $\sigma^i \leq \delta^{i+1}$ for some $\sigma^i \in \Sigma^i$, it follows from (6) and (4) that

$$\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^i)}{\sigma^i}. \tag{7}$$

Then

$$\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} \overset{(5)}{=} \frac{b(i + 1, \sigma^{i+1}) + d_i}{\sigma^{i+1} + \omega_i} = \frac{b(i + 1, \sigma^{i+1}) + \omega_i b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} \overset{(7)}{=} \frac{\sigma^{i+1} b(i, \sigma^i) + \omega_i b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} = \frac{b(i, \sigma^i)}{\sigma^i}.$$  

Hence $\sigma^{i+1} + \omega_i \in \Sigma^i$ and $b(i, \sigma^{i+1} + \omega_i) = b(i + 1, \sigma^{i+1}) + d_i = 1 - d_{S \cap P_i}$.

2. If $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$, $\delta^{i+1} + 1$ always belongs to $\Sigma^i$.

Suppose $S \cap P_i \notin \arg\max_{T \subset P_i \omega_T \geq q - \sigma^i} (1 - d_T)$ for all $\sigma^i \in \Sigma^i$. Then it must be the case that for any $\sigma^i$ either $\omega_{S \cap P_i} < q - \sigma^i$, or $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal.

Suppose $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $\delta^{i+1} + 1 \in \Sigma^i$, it follows from Lemma 6 that $\omega_{S \cap P_i} < q - \delta^i$. But then $\omega_{S \cap P_i} + \omega_{N \setminus P_i} = \omega_{S \cap P_i} + \omega_{N \setminus P_i} < q$, contradicting the assumption that $\omega_{S \cap P_i} \geq q - \sigma^{i+1}$.
Suppose $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i} < 1 - d_T$ for some $\sigma^i \in \Sigma^i$ and $T \subset P_i$ with $\omega_T \geq q - \sigma^i$.

If $\sigma^{i+1} + \omega_i > \delta^{i+1}$,

$$b(i, \sigma^{i+1} + \omega_i) = b(i, \sigma^i) > 1 - d_{S \cap P_i} = b(i + 1, \sigma^{i+1}) + d_i$$

contradicting (5).

If $\sigma^{i+1} + \omega_i \leq \delta^{i+1}$, $b(i, \sigma^{i+1} + \omega_i) \leq \frac{b(i, \tau^i)}{\tau^i}$. There are two possibilities:

- If $b(i, \tau^i) \geq 0$, it follows from (6) and (4) that
  $$\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \tau^i)}{\tau^i}$$
  (8)

  Then
  $$\frac{b(i, \tau^i)}{\tau^i}(\sigma^{i+1} + \omega_i) \geq b(i, \sigma^{i+1} + \omega_i) = b(i + 1, \sigma^{i+1}) + d_i = \frac{b(i, \tau^i)}{\tau^i}\sigma^{i+1} + b(i, \sigma^i) - (\sigma^i - \omega_i)\frac{b(i, \tau^i)}{\tau^i}$$

  implying $\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i)}{\sigma^i}$, thus $\frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \sigma^i)}{\sigma^i}$. It follows that $d_i^* = \omega_i \frac{b(i, \sigma^i)}{\sigma^i}$ and (from (8)) that $\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^i)}{\sigma^i}$. We can then follow the same steps as in case 1 to show that $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} = \frac{b(i, \sigma^i)}{\sigma^i}$. Hence $\sigma^{i+1} + \omega_i \in \Sigma^i$ and the result follows.

- If $b(i, \tau^i) < 0$, it follows from (6) and (4) that $\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = 0$. Then
  $$b(i, \sigma^{i+1} + \omega_i) = b(i + 1, \sigma^{i+1}) + d_i = b(i, \sigma^i).$$

  Hence $b(i, \sigma^i) = 1 - d_{S \cap P_i}$ and the result follows.

The next lemma shows that, if parties use $\Sigma^i$ to determine which coalition is formed, a party that demands more than $d_i^*$ will obtain a payoff of 0 in any SPE.
Lemma 9 Assume we are in $B(d, i + 1)$ and $d_i > d^*_i$.

a) If $b(i, \tau^i) \geq 0$ for some/all $\tau^i \in T^i$, then
   
   \[ i \notin S \text{ for all } S, \quad \arg\max_{T \subset P_{i+1}, \omega_T \geq q - \sigma^{i+1}} (1 - d_T) \text{ and all } \sigma^{i+1} \in \Sigma^{i+1}. \]

b) If $b(i, \tau^i) < 0$ for some/all $\tau^i \in T^i$, then every party obtains zero in any SPE of this subgame.

Proof. a) Let $\sigma^{i+1} \in \Sigma^{i+1}$ and $\tau^i \in T^i$. We need to prove that $b(i, \sigma^{i+1})$ exists and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i^i) - d_i$. This will be due to party $i + 1$ having the option of setting $\alpha = \sigma^i$ (if $\sigma^i \leq \delta^i + 1$) or $\alpha = \tau^i$ (if $\sigma^i > \delta^i + 1$).

We examine each case in turn:

1. If $\sigma^i \leq \delta^i + 1$, then $d_i > \frac{\omega_i b(i, \sigma^i)}{\sigma^i}$.

   Since $\sigma^i \leq \delta^i + 1$, $b(i + 1, \sigma^i)$ exists. Moreover, lemma 4b) implies

   \[ b(i + 1, \sigma^i) \geq b(i, \sigma^i). \tag{9} \]

   In principle, there are three possibilities for $\sigma^{i+1}$: either $\sigma^{i+1} < \gamma_0^0$, or $\sigma^{i+1} \geq \gamma_0^0$ and $b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i$, or $\sigma^{i+1} \geq \gamma_0^0$ and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$. We will show that the first two possibilities lead to a contradiction. In both cases, Lemma 4 implies

   \[ b(i + 1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \tag{10} \]

   From (10) we can deduce:

   \[
   \frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} < \frac{b(i, \sigma^{i+1} + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^{i+1}} \leq \frac{(\sigma^{i+1} + \omega_i)b(i, \sigma^i)}{\sigma^i} - \frac{\omega_i b(i, \sigma^i)}{\sigma^i} = \frac{b(i, \sigma^i)}{\sigma^i} \leq \frac{b(i + 1, \sigma^i)}{\sigma^i}. \]

26
which contradicts that \( \sigma^{i+1} \in \Sigma^{i+1} \). Thus, \( \sigma^{i+1} \geq \gamma_0^i \) (i.e. \( b(i, \sigma^{i+1}) \) does exist) and \( b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i \). We conclude then that \( i \notin S \) for all \( S \in \arg \max_{T \subset P_{\tau+1} \omega T \geq q - \sigma^{i+1}} (1 - d_T) \).

2. If \( \sigma^i > \delta^{i+1} \), then \( d_i > b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau_i} b(i, \tau_i) \).

Under Lemma 4b):

\[
b(i + 1, \tau_i) = \max \{ b(i, \tau_i), b(i, \tau_i + \omega_i) - d_i \} \geq b(i, \tau_i). \tag{11a}
\]

Suppose \( b(i, \sigma^{i+1}) \) does not exist (i.e. \( \sigma^{i+1} < \gamma_0^i \)), or \( b(i, \sigma^{i+1}) \) exists and \( b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i \). In both cases, under Lemma 4,

\[
b(i + 1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \tag{12}
\]

We will prove that (12) leads to a contradiction, so that \( b(i, \sigma^{i+1}) \) exits and \( b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i \), which implies \( i \notin S \) for all \( S \in \arg \max_{T \subset P_{\tau+1} \omega T \geq q - \sigma^{i+1}} (1 - d_T) \) as desired.

We have two cases:

- If \( \sigma^{i+1} + \omega_i \leq \delta^{i+1} \). Then \( \frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1}} \leq \frac{b(i, \tau_i)}{\tau_i} \) and

  \[
  \frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} < \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau_i} b(i, \tau_i)}{\sigma^{i+1}} \leq \frac{\sigma^{i+1} + \omega_i b(i, \tau_i) - \sigma^i b(i, \tau_i) + \frac{\sigma^i - \omega_i}{\tau_i} b(i, \tau_i)}{\sigma^{i+1}} = \frac{b(i, \tau_i) - (11a)}{\tau_i} \leq \frac{b(i + 1, \tau_i)}{\tau_i}
  \]

  which is a contradiction.

- If \( \sigma^{i+1} + \omega_i > \delta^{i+1} \), then under Corollary 2, \( b(i, \sigma^{i+1} + \omega_i) = b(i, \sigma^i) \). If \( b(i, \sigma^i) > 0, \sigma^i = \delta^{i+1} + 1 \) and \( \sigma^{i+1} + \omega_i \geq \sigma^i \), which
implies \((\sigma^i - \omega_i)/\sigma^{i+1} \leq 1\). If \(b(i, \sigma^i) = 0\), \(b(i, \tau^i) = 0\). Hence:
\[
\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} < \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} = \frac{\sigma^i - \omega_i}{\sigma^{i+1}} \frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i + 1, \tau^i)}{\tau^i}
\]
which is a contradiction.

b) Recall that we assumed \(b(i, \sigma^i) \geq 0\) for all \(\sigma^i \in \Sigma^i\). Thus, \(b(i, \tau^i) < 0\) for some \(\tau^i \in T^i\) implies \(\sigma^i > \delta^{i+1}\). Under Corollary 2, this means \(b(i, \sigma^i) = b(i, \delta^i)\). Let \(\alpha\) be such that \(\gamma^{i+1} \leq \alpha \leq \delta^{i+1}\). Under Lemma 4, we have two cases:

1. \(b(i + 1, \alpha) = b(i, \alpha + \omega_i) - d_i\). Then
\[
b(i + 1, \alpha) < b(i, \alpha + \omega_i) - b(i, \delta^i).
\]
Since \(\alpha + \omega_i \leq \delta^i\), \(b(i, \alpha + \omega_i) \leq b(i, \delta^i)\) and thus \(b(i + 1, \alpha) < 0\).

2. \(b(i + 1, \alpha) = b(i, \alpha)\). Then \(\gamma^{i+1}_0 \leq \alpha \leq \delta^{i+1}\) and
\[
\frac{b(i + 1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} < 0
\]
and thus \(b(i + 1, \alpha) < 0\).

Since \(b(i + 1, \alpha) < 0\) for all \(\alpha\), we conclude \(b(i + 1, \sigma^{i+1}) < 0\) for all \(\sigma^{i+1} \in \Sigma^{i+1}\) and thus by Lemma 5 all the parties get zero.

We will now construct a strategy profile for the parties and show it to be a SPE. In this strategy profile, parties will use the following tie-breaking rule: Given a party \(i\) and a set of coalitions \(S\), party \(i\) selects an element of \(S\) in the following way: First, select only the coalitions that contain the party with the highest index preceding \(i\) (party \(i - 1\), or, if party \(i - 1\) is in none
of the coalitions, party $i - 2$ etc.). If there are several coalitions containing this party, select the ones that contain the party with the second highest index, etc., until only one coalition is left. This procedure selects a coalition provided that $S \cap P_i \neq \emptyset$ for some $S \in S$. The role of the tie-breaking rule will be to ensure that parties moving before $i$ always have a best response (cf. example 1).

In $B(d, n)$, party $n$ forms a coalition $S \cup \{n\}$ with $S \in \arg \max_{T \subseteq P_n} (1 - d_T)$ after demanding $d_n = 1 - d_S$. If there is more than one possible choice of $S$, party $n$ uses the tie-breaking rule.

Let $i < n$ and assume we have defined the strategies for parties in $B(d, i + 1)$. In $B(d, i)$, party $i$ proceeds as follows:

1. If $\sigma_i > \omega_i$ for all $\sigma_i \in \Sigma_i$, party $i$ demands $d_i = d_i^*$ given as in (1).
2. If $\Sigma_i = \{\omega_i\}$, party $i$ forms coalition $S \cup \{i\}$ with $S \in \arg \min_{T \subseteq P_i} d_T$. If there is more than one possible choice of $S$, party $i$ uses the tie-breaking rule.
3. If $\{\omega_i\} \not\subseteq \Sigma_i$, party $i$ can anticipate the coalition $S^*$ that will be formed should it demand $d_i^*$ and its followers play the strategies we have defined.

   (a) If $i \notin S^*$, party $i$ forms coalition $S \cup \{i\}$ with $S \in \arg \min_{T \subseteq P_i} d_T$. If there is more than one possible $S$, party $i$ uses the tie-breaking rule.

   (b) If $i \in S^*$, party $i$ compares the coalitions $S \in \arg \max_{T \subseteq P_i} (1 - d_T)$ and $S^* \cap P_i$. Among them, party $i$ selects a coalition following the tie-breaking rule. If $S^*$ is chosen, party $i$ demands $d_i = d_i^*$ given as in (1). If $S \neq S^*$ is chosen, then party $i$ demands $1 - d_S = b(i, \omega_i) = d_i^*$ and forms coalition $S \cup \{i\}$.

**Proposition 1** The above strategies constitute a SPE for any $B(d, i)$.
Proof. We proceed by backwards induction on $i$. For $i = n$, its strategy is clearly optimal.

Assume now the result is true for $B(d,i+1)$ and moreover assume the following two conditions hold:

**Condition 1** The formed coalition satisfies

$$S \cap P_{i+1} \in \arg \max_{T \subseteq P_{i+1}, \omega_T \geq q-i_{i+1}} (1 - d_T)$$

for some $\sigma^{i+1} \in \Sigma^{i+1}$. (This condition holds trivially for $i + 1 = n$ because $\Sigma^n = \{\omega_n\}$).

**Condition 2** The above $S$ and $\sigma^{i+1}$ are such that $S \cap P_{i+1}$ is one of the most favorable sets for party $i$ (i.e. $i \notin S$ implies $i \notin T$ for all $T \in \arg \max_{T \subseteq P_{i+1}, \omega_T \geq q-i_{i+1}} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$). Among them, it is one of the most favorable to party $i - 1$, etc. (This condition holds for $i + 1 = n$ because $\Sigma^n = \{\omega_n\}$ and $n$ applies the tie-breaking rule).

We check that this remains true for $B(d,i)$. Let $\tau^i \in T^i$. We have two cases:

1. If $\sigma^i > \omega_i$ for all $\sigma^i \in \Sigma^i$, then it is straightforward to check that party $i$ obtains strictly less than $d^*_i$ by forming coalition. If $i$ demands $d^*_i$, there is a coalition $S \subseteq P_i$ such that $S \cup \{i\} \in \arg \max_{T \subseteq P_i, \omega_T \geq q-i_{i+1}} (1 - d_T)$ for $\sigma^{i+1} = \sigma^i - \omega_i \in \Sigma^{i+1}$ (Lemma 8b). The induction hypothesis (Conditions 1 and 2) implies that $d^*_i$ will be accepted. Assume party $i$ deviates by demanding $d_i > d^*_i$. If $b(i, \tau^i) \geq 0$, under Lemma 9a party $i$ does not belong to any coalition in $\arg \max_{T \subseteq P_{i+1}, \omega_T \geq q-i_{i+1}} (1 - d_T)$ for any $\sigma^{i+1} \in \Sigma^{i+1}$ and its final payoff is zero under the induction hypothesis (Condition 1). If $b(i, \tau^i) < 0$, under Lemma 9b, its final payoff is zero.
Moreover, Conditions 1 and 2 hold for \( i \). Condition 1 follows from Lemma 8 and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party \( j > i \) that eventually forms coalition.

2. If \( \omega_i \in \Sigma_i \), then \( 1 - d_S = b(i, \omega_i) = d_i^* \) for all \( S \in \arg\max_{T \subseteq P_i} (1 - d_T) \).

This means that if party \( i \) forms a winning coalition it obtains a final payoff of \( b(i, \omega_i) \). Suppose party \( i \) deviates and demands \( d_i > b(i, \omega_i) \). It is enough to check that \( i \notin S \) for all \( S \in \arg\max_{T \subseteq P_i + 1} (1 - d_T) \) and all \( \sigma^{i+1} \in \Sigma^{i+1} \). Under the induction hypothesis applied to Condition 1, this means that party \( i \) will not be included in any eventual winning coalition, and its final payoff will be zero, while the original strategy yields a nonnegative payoff.

For constant-sum homogeneous games it is always the case that \( \omega_i \leq \delta^{i+1} \), thus \( b(i + 1, \omega_i) \) is well defined. Under Lemma 4b,

\[
b(i + 1, \omega_i) = \max \{ b(i, \omega_i), b(i, 2\omega_i) - d_i \} \geq b(i, \omega_i) \quad (13)
\]

Suppose that \( i \in S \) for some \( S \in \arg\min_{T \subseteq P_i + 1} d_T \) and some \( \sigma^{i+1} \in \Sigma^{i+1} \). This means

\[
b(i + 1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i
\]

and hence

\[
\frac{b(i + 1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} < \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \omega_i)}{\sigma^{i+1}} \leq \frac{\sigma^{i+1} + \omega_i b(i, \omega_i) - b(i, \omega_i)}{\omega_i \sigma^{i+1}} = \frac{(\sigma^{i+1} + \omega_i) b(i, \omega_i) - \omega_i b(i, \omega_i)}{\omega_i \sigma^{i+1}} = b(i, \omega_i) \leq \frac{b(i + 1, \omega_i)}{\omega_i}
\]

31
which is a contradiction. This contradiction proves that $i \notin S$ for all $S \in \arg \min_{T \subseteq P_{i+1}} d_T$, as desired.

We now check that Conditions 1 and 2 hold for $i$. If party $i$ forms coalition, Condition 1 holds with $\sigma^i = \omega_i$, and Condition 2 holds because of the tie-breaking rule. If party $i$ demands $d^*_i$ so that $S^*$ is induced, it must be the case that $\{\omega_1\} \subsetneq \Sigma^i$. Hence, there exists $\sigma^i \in \Sigma^i$ with $\sigma^i > \omega_i$. Then, Condition 1 follows from Lemma 8c) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party that eventually forms coalition.

The next proposition will show uniqueness of equilibrium payoffs. Equilibrium strategies are not unique for some subgames. In subgames $B(d, i)$ where no coalition can be formed (i.e., $b(i, \sigma^i) < 0$), any demand vector is part of a SPE and equilibrium payoffs are always 0 for all parties. Multiplicity may also arise in subgames where a coalition can be formed but $d^*_i = 0$, as the following example illustrates.

**Example 4** Consider the game $[5; 3, 2, 2, 1, 1]$ and suppose $d_1 = d_2 = 1$. Equilibrium strategies at $B(d, 3)$ are not unique, but equilibrium payoffs are.

At $B(d, 3)$ we have $d^*_3 = 0$ and $\Sigma^3 = \{2, 3\}$. If we look at this subgame in isolation, several equilibrium outcomes are possible: coalition $\{1, 3\}$ (associated to $\sigma^3 = 2$), coalition $\{2, 3, 4\}$ or $\{2, 3, 5\}$ (associated to $\sigma^3 = 3$), a coalition like $\{2, 3, 4, 5\}$ (not a minimal winning coalition), coalition $\{1, 4, 5\}$ (which does not include party 3), or even no winning coalition at all. Intuitively, since the parties in $\{3, 4, 5\}$ cannot get a positive payoff, they are indifferent between all these situations. However, parties that have moved before are not indifferent. If we take into account that the strategies must be part of an equilibrium for all the subgames, and in particular for subgame
\( \mathbb{B}(d, 2) \), some of the equilibrium strategies at \( \mathbb{B}(d, 3) \) are not equilibrium strategies for \( \mathbb{B}(d, 2) \) and are discarded (cf. example 1). In particular, a coalition containing party 2 must be formed in order for party 2 to have a best response at \( \mathbb{B}(d, 2) \). Nevertheless, multiplicity remains: after party 2 sets \( d_2 = 1 \), there are three possible equilibrium coalitions: \{2, 3, 4\}, \{2, 3, 5\} and \{2, 3, 4, 5\}. Nevertheless, all equilibrium strategies lead to the same payoffs.

**Proposition 2** Assume we are in a SPE in \( \mathbb{B}(d, i) \). If \( b(i, \sigma^i) \geq 0 \) for some/all \( \sigma^i \in \Sigma^i \), party i’s payoff is \( d^*_i \) as defined in (1); otherwise party i’s payoff is zero.

**Proof.** We proceed by backwards induction on \( i \). We prove the following three hypotheses:

1. If \( b(i, \sigma^i) < 0 \), all parties get zero in every SPE of \( \mathbb{B}(d, i) \).

2. If \( b(i, \sigma^i) > 0 \), party i receives \( d^*_i > 0 \) in every SPE of \( \mathbb{B}(d, i) \) and the coalition that forms satisfies \( S \cap P_i \in \arg \max_{T \subseteq P_i} \omega_T \geq q - \sigma^i \) for some \( \sigma^i \in \Sigma^i \).

3. If \( b(i, \sigma^i) = 0 \),

   a) party i gets \( d^*_i = 0 \) in every SPE of \( \mathbb{B}(d, i) \);

   b) there is a SPE of \( \mathbb{B}(d, i) \) in which a winning coalition forms;

   c) if a winning coalition \( S \) forms, then \( S \cap P_i \in \arg \max_{T \subseteq P_i} \omega_T \geq q - \sigma^i \) for some \( \sigma^i \in \Sigma^i \).

The induction hypothesis holds for party \( n \). Now suppose it holds for party \( i + 1 \). Does it hold for party \( i \)?

1. If \( b(i, \sigma^i) < 0 \), all parties get zero (Lemma 5).
2. If $b(i, \sigma^i) > 0$, party $i$ cannot get more than $d^*_i$ by forming coalition. If party $i$ demands more than $d^*_i$ and $b(i, \tau^i) \geq 0$, we know from Lemma 9a) that $i \notin \arg \max_{T \subseteq P_i : \omega_T \geq q - \sigma^i} (1 - d_T)$ for all $\sigma^{i+1} \in \Sigma^{i+1}$. The induction hypothesis implies that party $i$ gets zero. If party $i$ demands more than $d^*_i$ and $b(i, \tau^i) < 0$, we know from Lemma 9b) that party $i$ gets zero.

Now we show that party $i$ can get at least $d^*_i$. This is immediate if $\omega_i \in \Sigma^i$. Suppose $\omega_i \notin \Sigma^i$. Since $b(i, \sigma^i) > 0$, we know $d^*_i > 0$. The value of $d^*_{i+1}$ induced by $d^*_i$ may be strictly positive or 0. Suppose party $i$ demands $d_i < d^*_i$. Then the corresponding value of $d^*_{i+1}$ is strictly positive. Under Lemma 8a), party $i$ belongs to all coalitions associated with some element of $\Sigma^{i+1}$, and the induction hypothesis for $d^*_{i+1} > 0$ implies that party $i$ gets $d_i$. Thus, the perfectness of the equilibrium implies that $d^*_i$ is accepted (otherwise, party $i$ would not have a best response).

Moreover, Lemma 8c), the induction hypothesis and the fact that $d^*_i$ is accepted imply that the coalition that forms satisfies $S \cap P_i \in \arg \max_{T \subseteq P_i : \omega_T \geq q - \sigma^i} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

3. If $b(i, \sigma^i) = 0$, then $d^*_i = 0$ and, moreover, $\alpha \in \Sigma^i$ if and only if $b(i, \alpha) = 0$.

a) It is trivial that party $i$ gets $d^*_i = 0$. If $d_i > d^*_i$, the induction hypothesis implies that no coalition to which party $i$ belongs will form.

b) There is an equilibrium of the subgame in which a coalition associated with $\sigma^i \in \Sigma^i$ forms. This is clearly the case for $\omega_i \in \Sigma^i$. Otherwise, it is optimal for party $i$ to demand $d^*_i = 0$. Then $b(i + 1, \sigma^{i+1}) = 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and the induction hypothesis implies that there is a SPE of $B(i, d)$ in which a winning coalition is formed.

c) Assume a winning coalition $S$ is formed with $S \cap P_i \notin \arg \max_{T \subseteq P_i : \omega_T \geq q - \sigma^i} (1 - d_T)$
for all $\sigma^i \in \Sigma^i$. This means that, for a given $\sigma^i \in \Sigma^i$, either $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal, or $\omega_{S \cap P_i} < q - \sigma^i$.

Assume first there exists $\sigma^i \in \Sigma^i$ such that $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal. Since $b(i, \sigma^i) = 0$, this means $d_{S \cap P_i} > 1$ and it cannot be optimal at any subgame to form $S$.

Assume now $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $b(i, \sigma^i) = 0$ and $b(i, \alpha)$ is nondecreasing in $\alpha$, $\delta^i \in \Sigma^i$; thus $\omega_{S \cap P_i} < q - \delta^i$. This means $\omega_{S \cap P_i} + \omega_{S \cap (N \setminus P_i)} < q$. Thus, $S$ is not a winning coalition.

\[\blacksquare\]

**Corollary 3** In any SPE, the coalition of Lemma 2 forms with each party demanding $d_i = \frac{\omega_i}{q}$.

**Proof.** Denote this coalition by $S^*$. Because of lemma 2, $S^* = P_{l+1}$ for some value of $l$. We can show $d_i = \frac{\omega_i}{q}$ for $i = 1, ..., l$ by induction on $i$.

Party 1 finds $\Sigma^1 = \{q\}$ and, since $q \leq \delta^2$ (due to the absence of veto players and the game being constant-sum) sets a demand $d^*_1 = \frac{\omega_1}{q}$. Given this demand, $q - \omega_1 \in \Sigma^2$.

Assume now $d_j = \frac{\omega_j}{q}$ for all $j \in P_l$, and $q - \omega_{P_l} \in \Sigma^i$. Then, since $q - \omega_{P_l} \leq \delta^{i+1}$,

$$d_i^* = \frac{\omega_i b(i, q - \omega_{P_l})}{q - \omega_{P_l}} = \frac{\omega_i \left(1 - \frac{\omega_{P_l}}{q}\right)}{q - \omega_{P_l}} = \frac{\omega_i}{q}$$

\[\blacksquare\]
References


