Strictly Monotonic Preferences on Continuum of Goods Commodity Spaces*

Carlos Hervés Beloso and P. K. Monteiro

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C. Hervés-Beloso
RGEA, Facultad de Económicas, Universidad de Vigo
Campus Universitario, 36310 Vigo, Spain

P.K. Monteiro
FGV-EPGE, Praia de Botafogo 190, sala 1103
22250-900 Rio de Janeiro, RJ, Brazil

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Abstract
We consider a set $K$ of differentiated commodities. A preference relation on the set of consumption plans is strictly monotonic whenever to consume more of at least one commodity is more preferred. It is an easy task to find examples of strictly monotonic preference relations when $K$ is finite or countable. However, it is not easy for spaces like $l^\infty([0,1])$, the space of bounded functions on the unit interval.

In this note we investigate the roots of this difficulty. We show that strictly monotonic preferences always exist. However, if $K$ is uncountable no such preference on $l^\infty(K)$ is continuous and none of them have a utility representation.

Keywords: utility representation, strictly monotonic preferences

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1 Introduction

This note is motivated by the lack of examples in the literature of pure strictly monotonic preferences in economies with a continuum of goods. For example, how would we picture a strictly monotonic preference relation in $l^\infty([0,1])$? Are there strictly monotonic utility functions in this space? Are there continuous ones? Indeed, we will show that there are strictly monotonic preference relations on any space of functions. The example will be similar to the lexicographic order. Thus, our example has no utility representation and is discontinuous. Can we find an example that is continuous? Even better, can we find a continuous utility function on $l^\infty(K)$ which is strictly monotonic? We will answer these questions negatively:

i) no strictly monotonic preference on $l^\infty(K)$ can be continuous in a linear topology if $K$ is uncountable;

ii) no such preference has a utility representation.

These results fully explain the lack of examples of pure strictly monotonic preferences in the literature. Our results imply that if we consider representable or continuous strictly monotonic preferences on a consumption space with a continuum of commodities, the consumption set should be, typically, a subset of the space of class of integrable functions, or the space of continuous functions. In these spaces to improve a consumption bundle it is necessary to consume more of uncountably many commodities.

2 Notation and assumptions

The notation is quite standard. We denote by $K$ the set of elementary commodities. The set of real functions on $K$ is denoted by $F(K)$. We use the standard partial order on $F(K)$. Thus, if $f,g \in F(K)$, we have that $f \succeq g$ if $f(k) \succeq g(k)$ for every $k \in K$. And we write $f > g$ if $f \succeq g$ and $f \neq g$. If $f \succeq g$ we define the interval $[g,f] = \{h \in F(K) ; g \leq h \leq f\}$.

The commodity space is a subspace of $F(K)$ and the consumption set, $X$, is, in any case, a subset of $F(K)$. A consumption plan $f \in X$ specifies an amount $f(k) \in \mathbb{R}$ of each commodity $k \in K$.

A preference relation $\succeq$ is a complete and transitive relation on $X$. Thus for every $f,g \in X$ either $f \succeq g$ or $g \succeq f$. Moreover if $f \succeq g \succeq h$ then $f \succeq h$. A preference relation has a utility representation (in short, is representable) if there is a function $U = U_\succeq : X \to \mathbb{R}$ such that $f \succeq g$ if and only if $U(f) \geq U(g)$. For $f,g \in X$ we write $f > g$ if $f \succeq g$ but $g \not\succeq f$. 

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It is quite easy to check that if \( U : X \rightarrow \mathbb{R} \) is a function then \( \succeq_U := \{(x, y) \in X \times X; U(x) \geq U(y)\} \) defines a complete and transitive preference relation on \( X \). Also \( x \succ_U y \) if and only if \( U(x) > U(y) \).

The preference relation on \( X \) is strictly monotonic if \( f, g \in X \) and \( f > g \) implies that \( f \succ g \). Let \( \tau \) be a topology on \( X \). The preference relation on \( (X, \tau) \) is continuous if \( \succeq \) is a closed subset of cartesian product \( X \times X \).

Let \( K \) be a set and \( \leq \) an order on \( K \). We say that \( \leq \) is a well-order in \( K \) if for every non-empty subset \( A \subset K \) there exists \( \min A \). The following theorem will be quite useful.

**Zermelo’s theorem** For every set \( K \) there is a well-ordering\(^1\) of \( K \).

### 3 Results

If \( K \) is countable it is quite easy to find strictly monotonic preferences on \( X \). For example, consider the case in which the consumption plans are bounded functions on \( K \) (sequences). Let \( a_n > 0 \) be such that \( \sum_{n=1}^{\infty} a_n < \infty \). The preference relation given by the function \( U(x) = \sum_{n=1}^{\infty} a_n x_n \) is strictly monotone, continuous\(^2\) and, obviously, representable by a utility function.

However next theorem shows that we cannot go much farther. In the proof we need to consider an order on \( K \). For example if \( K = [a, b] \) we take the usual order of the real numbers. More generally if \( a, b \in \mathbb{R}^n \) and \( K = \{(1 - t)a + tb; 0 \leq t \leq 1\} \) is the segment joining \( a \) and \( b \) we order \( K \) using the order on \([0, 1]\). Since Zermelo’s theorem says that every set has a well-ordering, we can suppose, without loss of generality, that \( K \) is ordered.

On the other hand, we will require the consumption set \( X \) to be rich enough in relation to \( K \). To be precise, let \( \chi_A \) denote the characteristic function of the set \( A \). That is, \( \chi_A(x) = 1 \) if \( x \in A \) and is 0 otherwise. The consumption plan \( \chi_A \) specifies the consumption of one unit of each commodity of the subset \( A \subset K \). We assume \((H.1)\)

\[(H.1) \text{ For any } A \subset K, \chi_A \in X\]

An example of commodity space that fulfils \((H.1)\) is the space of bounded functions on \( K \)

\[
B(K) = l^\infty(K) = \{f : K \rightarrow \mathbb{R}; \sup|f(x)|, x \in K < \infty\}.
\]

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\(^1\)This theorem was proved by E. Zermelo. For a proof see Kelley’s book page 33.

\(^2\)In the weak* topology and in the norm topology a fortiori.
**Theorem 1** Suppose that \((K, \leq)\) is an uncountable ordered set and that \(X\) fulfils \((H.1)\). Then every strictly monotonic preference relation on \(X\) is non-representable.

**Proof.** Let \(\succeq\) be a strictly monotonic preference relation on \(X\). Suppose that \(U : X \to \mathbb{R}\) represents \(\succeq\). Define\(^3\) for \(t \in K\) the functions \(x_t = \chi_{\{k \in K; k < t\}}\) and \(y_t = \chi_{\{k \in K; k \leq t\}}\). Since \(y_t > x_t\) we have that \(U(y_t) > U(x_t)\). Now, if \(t < s\) we have that \(x_s > y_t\) and therefore \(U(x_s) > U(y_t)\). In particular we conclude that the set of intervals \(\{I_t; t \in K\}\) where \(I_t = (U(x_t), U(y_t))\) is pairwise disjoint and uncountable. An impossibility. QED

**Remark 1** Actually, our proof only requires the existence of an uncountable subset \(K_0 \subset K\) such that the restriction of the characteristic functions \(x_t\) and \(y_t\) to \(K_0\) are in the consumption set \(X\) for all \(t \in K_0\).

Also, the same proof would work if instead of supposing that \(X\) fulfils \((H.1)\), we suppose that \(X\) contains an interval \([u, v]\) where \(u\) and \(v\) are such that \(v(k) - u(k) > 0\) for every \(k \in K\).

The theorem above would have no meaning if there are no strictly monotonic preference relations on \(X\). We fill this gap in the next theorem.

**Theorem 2** For any set \(K\) and for any consumption set \(X \subset F(K)\), there exists a strictly monotonic preference relation on \(X\).

**Proof.** Let \(\leq_K\) be a well-ordering of \(K\) and \(x, y \in X, x \neq y\). We denote \(k^* = \min\{k; x(k) \neq y(k)\}\). If \(x(k^*) > y(k^*)\) we define \(x \succ y\). Otherwise we define \(y \succ x\). It is a simple task to verify that \(\geq\) so defined is complete, transitive and strictly monotonic. QED

The preference relation defined above is a generalization of the lexicographic ordering which is discontinuous and non-representable (see Debreu (1954), footnote 1).

Our next objective is give an answer to the following question:

Is it possible to find a strictly monotonic preference on \(X\) which is continuous in a suitable topology?

For example if \(X\) is the positive cone of the space \(B(K)\) of bounded functions on \(K\) and \(\tau\) is such that \((B(K), \tau)\) is a Hausdorff topological vector space, can we find such a preference?

We will show that this is not possible.

\(^3\)The reader may suppose \(K = [a, b] \subset \mathbb{R}\) on a first reading.
In order to relate the continuity of a preference relation and the existence of a continuous utility representation, we remark that our commodity space, as a topological space, need not be second countable. Thus, we cannot appeal to Debreu’s representation theorem (1954). For nonseparable spaces the Monteiro’s representation theorem (1987), inspired in a result of Mas-Colell (1985), is very useful.

**Proposition** (Mas-Colell (1985)) Suppose that \((B(K), \tau)\) is a topological vector space. If \(\succeq\) is a monotonic and continuous preference relation, then \(\succeq\) restricted to \([0, \chi_K]\) has a utility representation.

**Proof.** Let \(\succeq' := \succeq \cap ([0, \chi_K] \times [0, \chi_K])\). That is, \(\succeq'\) is \(\succeq\) restricted to \([0, \chi_K]\). It is therefore continuous and, thus, \(\succeq'\) has a utility representation by Proposition 1, page 1044 of Mas-Colell (1985). (See also Corollary 2, page 150 of Monteiro (1987)). QED

As a consequence, we obtain

**Theorem 3** Let \(\tau\) be a linear topology on the commodity space and suppose that the consumption set \(X\) fulfil the assumptions of Theorem 1. Then, no strictly monotonic preference on \(X\) can be continuous on \((X, \tau)\).

**Proof.** If the preference relation is continuous, the restriction of this preference to \([0, \chi_K]\) will has a utility representation. However, this is in contradiction with Theorem 1. QED

In non-separable metric spaces there is always a non-representable continuous preference relation (M. Estévez and C. Hervés-Beloso 1995, page 306). Our results complement this nicely: It is not possible to strengthen the result by requiring strict monotonicity.\(^4\)

To make a counterpoint to the above results, let us consider the following example.

**Example**

There is a continuous strictly monotonic utility function on the space

\[
\ell_1([0, 1]) = \left\{ f : K \to \mathbb{R}; \sum_{t \in K} |f(t)| < \infty \right\}.
\]

To see this, let define \(U(f) = |f|_1 = \sum_{t \in K} |f(t)|\). This does not contradict theorem 1 since any \(f \in \ell_1([0, 1])\) is such that \(\{t; f(t) \neq 0\}\) is countable.

\(^4\)In spaces like \(B(K)\), of course.
We remark that in this consumption space no agent ever consumes an uncountable set of commodities. The example also shows the necessity of assumption $(H.1)$ in Theorem 1 and Theorem 3.

4 Conclusion

A preference is purely strictly monotonic whenever to consume more of at least one commodity is more preferred. Since we have considered a general scenario where this monotonicity is meaningful, we have proved that pure strictly monotonic preferences always exist. However, quite surprisingly, this strict monotonicity is incompatible with continuity and utility representation if there is a continuum of goods.

It is important to highlight that our incompatibility results do not apply neither to the case in which the commodity space is a space of the class of integrable functions nor to the space of continuous functions. In fact, the argument that leads to the proof of our theorem 1, requires the characteristic functions $x_t$ and $y_t$ to be different. However, as these functions only differ on $t$, both represent the same class of integrable functions on a $\sigma$-finite measure space. On the other hand, no consumption set in the space of continuous functions fulfils the assumption $(H.1)$.

We remark that in both kinds of commodity spaces, the strict monotonicity is not pure in the sense that $x \geq y$ and $x \neq y$ implies that in the consumption plan $x$ it is consumed more than in plan $y$ of uncountable many commodities.

References


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